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Finite Action in d5 Gauged Supergravity and Dilatonic Conformal Anomaly for Dual Quantum Field Theory

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ABSTRACT

Gauged supergravity (SG) with single scalar (dilaton) and arbitrary scalar potential is considered. Such dilatonic gravity describes special RG flows in extended SG where scalars lie in one-dimensional submanifold of total space. The surface counterterm and finite action for such gauged SG in three-, four- and five-dimensional asymptotically AdS space are derived. Using finite action and consistent gravitational stress tensor (local surface counterterm prescription) the regularized expressions for free energy, entropy and mass of d4 dilatonic AdS black hole are found. The same calculation is done within standard reference background subtraction.

The dilaton-dependent conformal anomaly from d3 and d5 gauged SGs is calculated using AdS/CFT correspondence. Such anomaly should correspond to two- and four-dimensional dual quantum field theory which is classically (not exactly) conformally invariant, respectively. The candidate c-functions from d3 and d5 SGs are suggested. These c-functions which have fixed points in asymptoticaly AdS region are expressed in terms of dilatonic potential and they are positively defined and monotonic for number of potentials.

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1 Introduction

AdS/CFT correspondence [1] may be realized in a sufficiently simple form as d5 gauged supergravity/boundary gauge theory correspondence. The reason is very simple: different versions of five-dimensional gauged SG (for example, $N = 8$ gauged SG [2] which contains 42 scalars and non-trivial scalar potential) could be obtained as compactification (reduction) of ten-dimensional IIB SG. Then, in practice it is enough to consider 5d gauged SG classical solutions (say, AdS-like backgrounds) in AdS/CFT set-up instead of the investigation of much more involved, non-linear equations of IIB SG. Moreover, such solutions describe RG flows in boundary gauge theory (for a very recent discussion of such flows see [3, 4, 32, 5, 6, 7, 8] and refs. therein). To simplify the situation in extended SG one can consider the symmetric (special) RG flows where scalars lie in one-dimensional submanifold of total space. Then, such theory is effectively described as d5 dilatonic gravity with non-trivial dilatonic potential. Nevertheless, it is still extremely difficult to make the explicit identification of deformed SG solution with the dual (non-conformal exactly) gauge theory. As a rule [4, 7], only indirect arguments may be suggested in such identification[4].

From another side, the fundamental holographic principle [9] in AdS/CFT form enriches the classical gravity itself (and here also classical gauged SG). Indeed, instead of the standard subtraction of reference background [10, 11] in making the gravitational action finite and the quasilocal stress tensor well-defined one introduces more elegant, local surface counterterm prescription [12]. Within it one adds the coordinate invariant functional of the intrinsic boundary geometry to gravitational action. Clearly, that does not modify the equations of motion. Moreover, this procedure has nice interpretation in terms of dual QFT as standard regularization. The specific choice of

[4]Such dual theory in massless case is, of course, classically conformally invariant and it has well-defined conformal anomaly. However, among the interacting theories only $\mathcal{N} = 4$ SYM is known to be exactly conformally invariant. Its conformal anomaly is not renormalized. For other, d4 QFTs there is breaking of conformal invariance due to radiative corrections which give contribution also to conformal anomaly. Hence, one can call such theories as non-conformal ones or not exactly conformally invariant. The conformal anomalies for such theories are explicitly unknown. Only for few simple theories (like scalar QED or gauge theory without fermions) the calculation of radiative corrections to conformal anomaly has been done up to two or three loops. It is a challenge to find exact conformal anomaly. Presumbly, only SG description may help to resolve this problem.

surface counterterm cancels the divergences of bulk gravitational action. As a by-product, it also defines the conformal anomaly of boundary QFT.

Local surface counterterm prescription has been successfully applied to construction of finite action and quasilocal stress tensor on asymptotically AdS space in Einstein gravity [12, 13, 14, 15, 16] and in higher derivative gravity [17]. Moreover, the generalization to asymptotically flat spaces is possible as it was first mentioned in ref.[18]. Surface counterterm has been found for domain-wall black holes in gauged SG in diverse dimensions [19]. However, actually only the case of asymptotically constant dilaton has been investigated there.

In the present paper we discuss the construction of finite action, consistent gravitational stress tensor and dilaton-dependent Weyl anomaly for boundary QFT (from bulk side) in three- and five-dimensional gauged supergravity with single scalar (dilaton) on asymptotically AdS background. Note that dilaton is not constant and the potential is chosen to be arbitrary. The implications of results for the study of RG flows in boundary QFT are presented, in particular, the candidate c-function is suggested.

The next section is devoted to the evaluation of Weyl anomaly from gauged supergravity with arbitrary dilatonic potential via AdS/CFT correspondence. We present explicit result for d3 and d5 gauged SGs. Such SG side conformal anomaly should correspond to dual QFT with broken conformal invariance in two and in four dimensions, respectively. The explicit form of d4 conformal anomaly takes few pages, so its lengthy dilaton-dependent coefficients are listed in Appendix. The comparison with similar AdS/CFT calculation of conformal anomaly in the same theory but with constant dilatonic potential is given. The candidates for c-function in two and four dimensions are proposed.

Section three is devoted to presentation of acceptable proposal for candidate c-fnction given in terms of dilatonic potential. It is shown that for numberof potentials such c-function is monotonic and positively defined. It has fixed point in asymptotically AdS region. The comparison with other c-functions is given.

In section four we construct surface counterterms for d3 and d5 gauged SGs. As a result, the gravitational action in asymptotically AdS space is finite. On the same time, the gravitational stress tensor around such space is well defined. It is interesting that conformal anomaly defined in second section directly follows from the gravitational stress tensor with account of

surface terms.

Section five is devoted to the application of finite gravitational action found in previous section in the calculation of thermodynamical quantities in dilatonic AdS black hole. The dilatonic AdS black hole is constructed approximately, using the perturbations around constant dilaton AdS black hole. The entropy, mass and free energy of such black hole are found using the local surface counterterm prescription to regularize these quantities. The comparison is done with the case when standard prescription: regularization with reference background is used. The explicit regularization dependence of the result is mentioned. Finally, in the Discussion the summary of results is presented and some open problems are mentioned.

2 Weyl anomaly for gauged supergravity with general dilaton potential

In the present section the derivation of dilaton-dependent Weyl anomaly from gauged SG will be given. As we note in section 4 this derivation can be made also from the definition of finite action in asymptotically AdS space.

We start from the bulk action of $d+1$-dimensional dilatonic gravity with the potential Φ

$$S=\frac{1}{16\pi G}\int_{M_{d+1}} d^{d+1}x\sqrt{-\hat{G}}\left\{\hat{R}+X(\phi)(\hat{\nabla}\phi)^2+Y(\phi)\hat{\Delta}\phi+\Phi(\phi)+4\lambda^2\right\}\ . \tag{1}$$

Here M_{d+1} is $d+1$ dimensional manifold whose boundary is d dimensional manifold M_d and we choose $\Phi(0)=0$. Such action corresponds to (bosonic sector) of gauged SG with single scalar (special RG flow). In other words, one considers RG flow in extended SG when scalars lie in one-dimensional submanifold of complete scalars space. Note also that classical vacuum stability restricts the form of dilaton potential [20]. As well-known, we also need to add the surface terms [10] to the bulk action in order to have well-defined variational principle. At the moment, for the purpose of calculation of Weyl anomaly (via AdS/CFT correspondence) the surface terms are irrelevant. The equations of motion given by variation of (1) with respect to ϕ and $G^{\mu\nu}$ are

$$0 \ = \ -\sqrt{-\hat{G}}\Phi'(\phi)-\sqrt{-\hat{G}}V'(\phi)\hat{G}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

$$+2\partial_\mu \left(\sqrt{-\hat{G}}\hat{G}^{\mu\nu}V(\phi)\partial_\nu\phi\right) \tag{2}$$

$$0 = \frac{1}{d-1}\hat{G}_{\mu\nu}\left(\Phi(\phi)+\frac{d(d-1)}{l^2}\right)+\hat{R}_{\mu\nu}+V(\phi)\partial_\mu\phi\partial_\nu\phi \ . \tag{3}$$

Here

$$V(\phi) \equiv X(\phi) - Y'(\phi) \ . \tag{4}$$

We choose the metric $\hat{G}_{\mu\nu}$ on M_{d+1} and the metric $\hat{g}_{\mu\nu}$ on M_d in the following form

$$ds^2 \equiv \hat{G}_{\mu\nu}dx^\mu dx^\nu = \frac{l^2}{4}\rho^{-2}d\rho d\rho + \sum_{i=1}^{d}\hat{g}_{ij}dx^i dx^j \ , \quad \hat{g}_{ij} = \rho^{-1}g_{ij} \ . \tag{5}$$

Here l is related with λ^2 by $4\lambda^2 = d(d-1)/l^2$. If $g_{ij} = \eta_{ij}$, the boundary of AdS lies at $\rho = 0$. We follow to method of calculation of conformal anomaly as it was done in refs.[21, 22] where dilatonic gravity with constant dilaton potential has been considered. Part of results of this section concerning Weyl anomaly with no dilaton derivatives has been presented already in letter [23].

The action (1) diverges in general since it contains the infinite volume integration on M_{d+1}. The action is regularized by introducing the infrared cutoff ϵ and replacing

$$\int d^{d+1}x \to \int d^dx \int_\epsilon d\rho \ , \quad \int_{M_d} d^dx\Big(\cdots\Big) \to \int d^dx \left(\cdots\right)\Big|_{\rho=\epsilon} \ . \tag{6}$$

We also expand g_{ij} and ϕ with respect to ρ:

$$g_{ij} = g_{(0)ij} + \rho g_{(1)ij} + \rho^2 g_{(2)ij} + \cdots \ , \quad \phi = \phi_{(0)} + \rho\phi_{(1)} + \rho^2\phi_{(2)} + \cdots \ . \tag{7}$$

Then the action is also expanded as a power series on ρ. The subtraction of the terms proportional to the inverse power of ϵ does not break the invariance under the scale transformation $\delta g_{\mu\nu} = 2\delta\sigma g_{\mu\nu}$ and $\delta\epsilon = 2\delta\sigma\epsilon$. When d is even, however, the term proportional to $\ln\epsilon$ appears. This term is not invariant under the scale transformation and the subtraction of the $\ln\epsilon$ term breaks the invariance. The variation of the $\ln\epsilon$ term under the scale transformation is finite when $\epsilon \to 0$ and should be canceled by the variation of the finite term (which does not depend on ϵ) in the action since the original action (1) is invariant under the scale transformation. Therefore the $\ln\epsilon$ term

$S_{\rm ln}$ gives the Weyl anomaly T of the action renormalized by the subtraction of the terms which diverge when $\epsilon \to 0$ (d=4)

$$S_{\rm ln} = -\frac{1}{2}\int d^4x\sqrt{-g}T \ . \tag{8}$$

The conformal anomaly can be also obtained from the surface counterterms, which is discussed in Section 4.

First we consider the case of $d = 2$, i.e. three-dimensional gauged SG. The anomaly term $S_{\rm ln}$ proportional to lnϵ in the action is

$$\begin{aligned} S_{\rm ln} = -\frac{1}{16\pi G}\frac{l}{2}\int d^2x\sqrt{-g_{(0)}}\Big\{ & R_{(0)} + X(\phi_{(0)})(\nabla\phi_{(0)})^2 + Y(\phi_{(0)})\Delta\phi_{(0)} \\ & +\phi_{(1)}\Phi'(\phi_{(0)}) + \frac{1}{2}g_{(0)}^{ij}g_{(1)ij}\Phi(\phi_{(0)})\Big\} . \end{aligned} \tag{9}$$

The terms proportional to ρ^0 with $\mu, \nu = i, j$ in (3) lead to $g_{(1)ij}$ in terms of $g_{(0)ij}$ and $\phi_{(1)}$.

$$\begin{aligned} g_{(1)ij} &= \Big[-R_{(0)ij} - V(\phi_{(0)})\partial_i\phi_{(0)}\partial_j\phi_{(0)} - g_{(0)ij}\Phi'(\phi_{(0)})\phi_{(1)} \\ &\quad +\frac{g_{(0)ij}}{l^2}\left\{2\Phi'(\phi_{(0)})\phi_{(1)} + R_{(0)} + V(\phi_{(0)})g_{(0)}^{kl}\partial_k\phi_{(0)}\partial_l\phi_{(0)}\right\} \\ &\quad \times\left(\Phi(\phi_{(0)}) + \frac{2}{l^2}\right)^{-1}\Big] \times \Phi(\phi_{(0)})^{-1} \end{aligned} \tag{10}$$

In the equation (2), the terms proportional to ρ^{-1} lead to $\phi_{(1)}$ as following.

$$\begin{aligned} \phi_{(1)} &= \Big[V'(\phi_{(0)})g_{(0)}^{ij}\partial_i\phi_{(0)}\partial_j\phi_{(0)} + 2\frac{V(\phi_{(0)})}{\sqrt{-g_{(0)}}}\partial_i\left(\sqrt{-g_{(0)}}g_{(0)}^{ij}\partial_j\phi_{(0)}\right) \\ &\quad +\frac{1}{2}\Phi'(\phi_{(0)})\left(\Phi(\phi_{(0)}) + \frac{2}{l^2}\right)^{-1}\{R_{(0)} + V(\phi_{(0)})g_{(0)}^{ij}\partial_i\phi_{(0)}\partial_j\phi_{(0)}\}\Big] \\ &\quad \times\left(\Phi''(\phi_{(0)}) - \Phi'(\phi_{(0)})^2\left(\Phi(\phi_{(0)}) + \frac{2}{l^2}\right)^{-1}\right)^{-1} \end{aligned} \tag{11}$$

Then anomaly term takes the following form using (10), (11)

$$T = \frac{1}{8\pi G}\frac{l}{2}\Big\{R_{(0)} + X(\phi_{(0)})(\nabla\phi_{(0)})^2 + Y(\phi_{(0)})\Delta\phi_{(0)}$$

$$
\begin{aligned}
&+\frac{1}{2}\left\{\frac{2\Phi'(\phi_{(0)})}{l^2}\left(\Phi''(\phi_{(0)})\left(\Phi(\phi_{(0)})+\frac{2}{l^2}\right)-\Phi'(\phi_{(0)})^2\right)^{-1}-\Phi(\phi_{(0)})\right\} \\
&\times\left(R_{(0)}+V(\phi_{(0)})g_{(0)}^{ij}\partial_i\phi_{(0)}\partial_j\phi_{(0)}\right)\left(\Phi(\phi_{(0)})+\frac{2}{l^2}\right)^{-1} \\
&+\frac{2\Phi'(\phi_{(0)})}{l^2}\left(\Phi''(\phi_{(0)})\left(\Phi(\phi_{(0)})+\frac{2}{l^2}\right)-\Phi'(\phi_{(0)})^2\right)^{-1} \\
&\times\left(V'(\phi_{(0)})g_{(0)}^{ij}\partial_i\phi_{(0)}\partial_j\phi_{(0)}+2\frac{V(\phi_{(0)})}{\sqrt{-g_{(0)}}}\partial_i\left(\sqrt{-g_{(0)}}g_{(0)}^{ij}\partial_j\phi_{(0)}\right)\right)\Bigg\}\ . \qquad (12)
\end{aligned}
$$

For $\Phi(\phi)=0$ case, the central charge of two-dimensional conformal field theory is defined by the coefficient of R. Then it might be natural to introduce the candidate c-function c for the case when the conformal symmetry is broken by the deformation in the following way :

$$
\begin{aligned}
c &= \frac{3}{2G}\left[l+\frac{l}{2}\left\{\frac{2\Phi'(\phi_{(0)})}{l^2}\left(\Phi''(\phi_{(0)})\left(\Phi(\phi_{(0)})\right.\right.\right.\right. \\
&\qquad \left.\left.\left.\left.+\frac{2}{l^2}\right)-\Phi'(\phi_{(0)})^2\right)^{-1}-\Phi(\phi_{(0)})\right\}\times\left(\Phi(\phi_{(0)})+\frac{2}{l^2}\right)^{-1}\right]\ . \qquad (13)
\end{aligned}
$$

Comparing this with radiatively-corrected c-function of boundary QFT (AdS_3/CFT_2) may help in correct bulk description of such theory. Clearly, that in the regions (or for potentials) where such candidate c-function is singular or not monotonic it cannot be the acceptable c-function. Presumbly, the appearence of such regions indicates to the breaking of SG description.

Four-dimensional case is more interesting but also much more involved. The anomaly terms which proportional to $\ln\epsilon$ are

$$
\begin{aligned}
S_{\rm ln} &= \frac{1}{16\pi G}\int d^4x\sqrt{-g_{(0)}}\left[\frac{-1}{2l}g_{(0)}^{ij}g_{(0)}^{kl}\left(g_{(1)ij}g_{(1)kl}-g_{(1)ik}g_{(1)jl}\right)\right. \\
&+\frac{l}{2}\left(R_{(0)}^{ij}-\frac{1}{2}g_{(0)}^{ij}R_{(0)}\right)g_{(1)ij} \\
&-\frac{2}{l}V(\phi_{(0)})\phi_{(1)}^2+\frac{l}{2}V'(\phi_{(0)})\phi_{(1)}g_{(0)}^{ij}\partial_i\phi_{(0)}\partial_j\phi_{(0)} \\
&+lV(\phi_{(0)})\phi_{(1)}\frac{1}{\sqrt{-g_{(0)}}}\partial_i\left(\sqrt{-g_{(0)}}g_{(0)}^{ij}\partial_j\phi_{(0)}\right) \\
&+\frac{l}{2}V(\phi_{(0)})\left(g_{(0)}^{ik}g_{(0)}^{jl}g_{(1)kl}-\frac{1}{2}g_{(0)}^{kl}g_{(1)kl}g_{(0)}^{ij}\right)\partial_i\phi_{(0)}\partial_j\phi_{(0)} \qquad (14)
\end{aligned}
$$

$$
-\frac{l}{2}\left(\frac{1}{2}g_{(0)}^{ij}g_{(2)ij}-\frac{1}{4}g_{(0)}^{ij}g_{(0)}^{kl}g_{(1)ik}g_{(1)jl}+\frac{1}{8}(g_{(0)}^{ij}g_{(1)ij})^2\right)\Phi(\phi_{(0)})
$$
$$
\left.-\frac{l}{2}\left(\Phi'(\phi_{(0)})\phi_{(2)}+\frac{1}{2}\Phi''(\phi_{(0)})\phi_{(1)}^2+\frac{1}{2}g_{(0)}^{kl}g_{(1)kl}\Phi'(\phi_{(0)})\phi_{(1)}\right)\right] \ .
$$

The terms proportional to ρ^0 with $\mu,\nu=i,j$ in the equation of the motion (3) lead to $g_{(1)ij}$ in terms of $g_{(0)ij}$ and $\phi_{(1)}$.

$$
\begin{aligned}
g_{(1)ij} &= \Big[-R_{(0)ij}-V(\phi_{(0)})\partial_i\phi_{(0)}\partial_j\phi_{(0)}-\frac{1}{3}g_{(0)ij}\Phi'(\phi_{(0)})\phi_{(1)} \\
&\quad +\frac{g_{(0)ij}}{l^2}\left\{\frac{4}{3}\Phi'(\phi_{(0)})\phi_{(1)}+R_{(0)}+V(\phi_{(0)})g_{(0)}^{kl}\partial_k\phi_{(0)}\partial_l\phi_{(0)}\right\} \\
&\quad \times\left(\frac{1}{3}\Phi(\phi_{(0)})+\frac{6}{l^2}\right)^{-1}\Big]\times\left(\frac{1}{3}\Phi(\phi_{(0)})+\frac{2}{l^2}\right)^{-1} \ .
\end{aligned} \tag{15}
$$

In the equation (2), the terms proportional to ρ^{-2} lead to $\phi_{(1)}$ as follows:

$$
\begin{aligned}
\phi_{(1)} &= \Big[V'(\phi_{(0)})g_{(0)}^{ij}\partial_i\phi_{(0)}\partial_j\phi_{(0)}+2\frac{V(\phi_{(0)})}{\sqrt{-g_{(0)}}}\partial_i\left(\sqrt{-g_{(0)}}g_{(0)}^{ij}\partial_j\phi_{(0)}\right) \\
&\quad +\frac{1}{2}\Phi'(\phi_{(0)})\left(\frac{1}{3}\Phi(\phi_{(0)})+\frac{6}{l^2}\right)^{-1}\left\{R_{(0)}+V(\phi_{(0)})g_{(0)}^{ij}\partial_i\phi_{(0)}\partial_j\phi_{(0)}\right\}\Big] \\
&\quad \times\left(\frac{8V(\phi_{(0)})}{l^2}+\Phi''(\phi_{(0)})-\frac{2}{3}\Phi'(\phi_{(0)})^2\left(\frac{1}{3}\Phi(\phi_{(0)})+\frac{6}{l^2}\right)^{-1}\right)^{-1} .
\end{aligned} \tag{16}
$$

In the equation (3), the terms proportional to ρ^1 with $\mu,\nu=i,j$ lead to $g_{(2)ij}$.

$$
\begin{aligned}
g_{(2)ij} &= \Big[-\frac{1}{3}\left\{g_{(1)ij}\Phi'(\phi_{(0)})\phi_{(1)}+g_{(0)ij}(\Phi'(\phi_{(0)})\phi_{(2)}+\frac{1}{2}\Phi''(\phi_{(0)})\phi_{(1)}^2)\right\} \\
&\quad -\frac{2}{l^2}g_{(0)}^{kl}g_{(1)ki}g_{(1)lj}+\frac{1}{l^2}g_{(0)}^{km}g_{(0)}^{nl}g_{(1)mn}g_{(1)kl}g_{(0)ij} \\
&\quad -\frac{2}{l^2}g_{(0)ij}\left(\frac{1}{3}\Phi(\phi_{(0)})+\frac{8}{l^2}\right)^{-1}\times\Big\{\frac{2}{l^2}g_{(0)}^{mn}g_{(0)}^{kl}g_{(1)km}g_{(1)ln} \\
&\quad -\frac{4}{3}\left(\Phi'(\phi_{(0)})\phi_{(2)}+\frac{1}{2}\Phi''(\phi_{(0)})\phi_{(1)}^2\right)-\frac{1}{3}g_{(0)}^{ij}g_{(1)ij}\Phi'(\phi_{(0)})\phi_{(1)} \\
&\quad +V'(\phi_{(0)})\phi_{(1)}g_{(0)}^{ij}\partial_i\phi_{(0)}\partial_j\phi_{(0)}+\frac{2V(\phi_{(0)})\phi_{(1)}}{\sqrt{-g_{(0)}}}\partial_i\left(\sqrt{-g_{(0)}}g_{(0)}^{ij}\partial_j\phi_{(0)}\right)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+V'(\phi_{(0)})\phi_{(1)}\partial_i\phi_{(0)}\partial_j\phi_{(0)} + 2V(\phi_{(0)})\phi_{(1)}\partial_i\partial_j\phi_{(0)}\Big] \\
&\times\left(\frac{1}{3}\Phi(\phi_{(0)})\right)^{-1} .
\end{aligned} \tag{17}
$$

And the terms proportional to ρ^{-1} in the equation (2), lead to $\phi_{(2)}$ as follows:

$$
\begin{aligned}
\phi_{(2)} \;=\; & \Big[V''(\phi_{(0)})\phi_{(1)}g_{(0)}^{ij}\partial_i\phi_{(0)}\partial_j\phi_{(0)} \\
& +V'(\phi_{(0)})\left(g_{(0)}^{ik}g_{(0)}^{jl} - \frac{1}{2}g_{(0)}^{ij}g_{(0)}^{kl}\right)g_{(1)kl}\partial_i\phi_{(0)}\partial_j\phi_{(0)} \\
& +\frac{2V'(\phi_{(0)})\phi_{(1)}}{\sqrt{-g_{(0)}}}\partial_i\left(\sqrt{-g_{(0)}}g_{(0)}^{ij}\partial_j\phi_{(0)}\right) \\
& -\frac{4}{l^2}V'_{(0)}\phi_{(1)}^2 - \frac{1}{2}\Phi'''(\phi_{(0)})\phi_{(1)}^2 - \frac{1}{2}g_{(0)}^{kl}g_{(1)kl}\Phi''(\phi_{(0)})\phi_{(1)} \\
& -\left(\frac{-1}{4}g_{(0)}^{ij}g_{(0)}^{kl}g_{(1)ik}g_{(1)jl} + \frac{1}{8}(g_{(0)}^{ij}g_{(1)ij})^2\right)\Phi'(\phi_{(0)}) \\
& -\frac{1}{2}\Phi'(\phi_{(0)})\left(\frac{1}{3}\Phi(\phi_{(0)}) + \frac{8}{l^2}\right)^{-1} \times \Big\{\frac{2}{l^2}g_{(0)}^{mn}g_{(0)}^{kl}g_{(1)km}g_{(1)ln} \\
& -\frac{2}{3}\Phi''(\phi_{(0)})\phi_{(1)}^2 - \frac{1}{3}g_{(0)}^{ij}g_{(1)ij}\Phi'(\phi_{(0)})\phi_{(1)} \\
& +V'(\phi_{(0)})\phi_{(1)}g_{(0)}^{ij}\partial_i\phi_{(0)}\partial_j\phi_{(0)} + \frac{2V(\phi_{(0)})\phi_{(1)}}{\sqrt{-g_{(0)}}}\partial_i\left(\sqrt{-g_{(0)}}g_{(0)}^{ij}\partial_j\phi_{(0)}\right)\Big\}\Big] \\
& \times\left(\Phi''(\phi_{(0)}) - \frac{2}{3}\Phi'(\phi_{(0)})^2\left(\frac{1}{3}\Phi(\phi_{(0)}) + \frac{8}{l^2}\right)^{-1}\right)^{-1}
\end{aligned} \tag{18}
$$

Then we can get the anomaly (14) in terms of $g_{(0)ij}$ and $\phi_{(0)}$, which are boundary values of metric and dilaton respectively by using (15), (16), (17), (18). In the following, we choose $l = 1$, denote $\Phi(\phi_{(0)})$ by Φ and abbreviate the index (0) for the simplicity. Then substituting (16) into (15), we obtain

$$
\begin{aligned}
g_{(1)ij} \;=\; & \tilde{c}_1R_{ij} + \tilde{c}_2g_{ij}R + \tilde{c}_3g_{ij}g^{kl}\partial_k\phi\partial_l\phi \\
& +\tilde{c}_4g_{ij}\frac{\partial_k}{\sqrt{-g}}\left(\sqrt{-g}g^{kl}\partial_l\phi\right) + \tilde{c}_5\partial_i\phi\partial_j\phi .
\end{aligned} \tag{19}
$$

The explicit form of $\tilde{c}_1$, $\tilde{c}_2$, $\cdots$ $\tilde{c}_5$ is given in Appendix A. Further, substituting (16) and (19) into (18), one gets

$$
\phi_{(2)} = d_1R^2 + d_2R_{ij}R^{ij} + d_3R^{ij}\partial_i\phi\partial_j\phi
$$

$$
\begin{aligned}
&+d_4 R g^{ij}\partial_i\phi\partial_j\phi + d_5 R\frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}g^{ij}\partial_j\phi) \\
&+d_6(g^{ij}\partial_i\phi\partial_j\phi)^2 + d_7\left(\frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}g^{ij}\partial_j\phi)\right)^2 \\
&+d_8 g^{kl}\partial_k\phi\partial_l\phi\frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}g^{ij}\partial_j\phi)\ . \qquad (20)
\end{aligned}
$$

Here, the explicit form of $d_1, \cdots d_8$ is given in Appendix A. Substituting (16), (19) and (20) into (17), one gets

$$
\begin{aligned}
g^{ij}g_{(2)ij} \;=\;& f_1 R^2 + f_2 R_{ij}R^{ij} + f_3 R^{ij}\partial_i\phi\partial_j\phi \\
&+f_4 R g^{ij}\partial_i\phi\partial_j\phi + f_5 R\frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}g^{ij}\partial_j\phi) \\
&+f_6(g^{ij}\partial_i\phi\partial_j\phi)^2 + f_7\left(\frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}g^{ij}\partial_j\phi)\right)^2 \\
&+f_8 g^{kl}\partial_k\phi\partial_l\phi\frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}g^{ij}\partial_j\phi)\ . \qquad (21)
\end{aligned}
$$

Again, the explicit form of very complicated functions $f_1, \cdots f_8$ is given in Appendix A. Finally substituting (16), (19), (20) and (21) into the expression for the anomaly (14), we obtain,

$$
\begin{aligned}
T \;=\;& -\frac{1}{8\pi G}\Big[h_1 R^2 + h_2 R_{ij}R^{ij} + h_3 R^{ij}\partial_i\phi\partial_j\phi \\
&+h_4 R g^{ij}\partial_i\phi\partial_j\phi + h_5 R\frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}g^{ij}\partial_j\phi) \\
&+h_6(g^{ij}\partial_i\phi\partial_j\phi)^2 + h_7\left(\frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}g^{ij}\partial_j\phi)\right)^2 \\
&+h_8 g^{kl}\partial_k\phi\partial_l\phi\frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}g^{ij}\partial_j\phi)\Big]\ . \qquad (22)
\end{aligned}
$$

Here

$$
\begin{aligned}
h_1 \;=\;& \Big[3\ \Big\{(24 - 10\ \Phi)\ \Phi'^6 \\
&+(62208 + 22464\ \Phi + 2196\ \Phi^2 + 72\ \Phi^3 + \Phi^4)\ \Phi''\ (\Phi'' + 8\ V)^2 \\
&+2\ \Phi'^4\ \Big\{(108 + 162\ \Phi + 7\ \Phi^2)\ \Phi'' + 72\ (-8 + 14\ \Phi + \Phi^2)\ V\Big\}
\end{aligned}
$$

$$
\begin{aligned}
& -2\ \Phi'^2\ \Big\{(6912+2736\ \Phi+192\ \Phi^2+\Phi^3)\ \Phi''^2 \\
& +4\ (11232+6156\ \Phi+552\ \Phi^2+13\ \Phi^3)\ \Phi''\ V \\
& +32\ (-2592+468\ \Phi+96\ \Phi^2+5\ \Phi^3)\ V^2\Big\} \\
& -3\ (-24+\Phi)\ (6+\Phi)^2\ \Phi'^3\ (\Phi'''+8\ V')\Big\}\Big]\Big/ \\
& \Big[16\ (6+\Phi)^2\ \Big\{-2\ \Phi'^2+(24+\Phi)\ \Phi''\Big\}\ \Big\{-2\ \Phi'^2 \\
& +(18+\Phi)\ (\Phi''+8\ V)\Big\}^2\Big] \\
h_2 \ = \ & -\frac{3\ \{(12-5\ \Phi)\ \Phi'^2+(288+72\ \Phi+\Phi^2)\ \Phi''\}}{8\ (6+\Phi)^2\ \{-2\ \Phi'^2+(24+\Phi)\ \Phi''\}}\ . \qquad (23)
\end{aligned}
$$

We also give the explicit forms of $h_3,\ \cdots\ h_8$ in Appendix A. Thus, we found the complete Weyl anomaly from bulk side. This expression which should describe dual d4 QFT of QCD type, with broken SUSY looks really complicated. The interesting remark is that Weyl anomaly is not integrable in general. In other words, it is impossible to construct the anomaly induced action. This is not strange, as it is usual situation for conformal anomaly when radiative corrections are taken into account.

In case of the dilaton gravity in [21] corresponding to $\Phi = 0$ (or more generally in case that the axion is included [24] as in [22]), we have the following expression:

$$
\begin{aligned}
T \ = \ & \frac{l^3}{8\pi G}\int d^4x\sqrt{-g_{(0)}}\left[\frac{1}{8}R_{(0)ij}R_{(0)}^{ij}-\frac{1}{24}R_{(0)}^2\right. \\
& -\frac{1}{2}R_{(0)}^{ij}\partial_i\varphi_{(0)}\partial_j\varphi_{(0)}+\frac{1}{6}R_{(0)}g_{(0)}^{ij}\partial_i\varphi_{(0)}\partial_j\varphi_{(0)} \\
& \left.+\frac{1}{4}\left\{\frac{1}{\sqrt{-g_{(0)}}}\partial_i\left(\sqrt{-g_{(0)}}g_{(0)}^{ij}\partial_j\varphi_{(0)}\right)\right\}^2+\frac{1}{3}\left(g_{(0)}^{ij}\partial_i\varphi_{(0)}\partial_j\varphi_{(0)}\right)^2\right] \qquad (24)
\end{aligned}
$$

Here φ can be regarded as dilaton. In the limit of $\Phi \to 0$, we obtain

$$
\begin{aligned}
h_1 \ & \to \ \frac{3\cdot 62208\Phi''(8V)^2}{16\cdot 6^2\cdot 24\cdot 18\Phi''(8V)^2}=\frac{1}{24} \\
h_2 \ & \to \ -\frac{3\cdot 288\Phi''}{8\cdot 6^2\cdot 24\Phi''}=-\frac{1}{8} \\
h_3 \ & \to \ -\frac{3\cdot 288(\Phi''V-\Phi'V')}{4\cdot 6^2\cdot 24\Phi''}=-\frac{1}{4}\frac{(\Phi''V-\Phi'V')}{\Phi''}
\end{aligned}
$$

$$
\begin{aligned}
h_4 &\rightarrow \frac{3\cdot 62208\Phi''V(8V)^2+6\Phi'\cdot 384\cdot(-5184)\cdot V^2V'}{8\cdot 6^2\cdot 24\Phi''\cdot(18\cdot 8V)^2}=\frac{1}{12}\frac{(\Phi''V-\Phi'V')}{\Phi''} \\
h_5 &\rightarrow 0 \\
h_6 &\rightarrow \Big\{-\Phi''\cdot 64V\cdot\Big(373248V^3-139968V'^2\Big) \\
&\qquad +2\cdot 6\Phi'V'\cdot(-2)\cdot(-432)\cdot\Big(4608V^3+864V'^2-1728VV''\Big)\Big\} \\
&\qquad /16\cdot 6^2\cdot 24\Phi''\cdot(18\cdot 8V)^2 \\
&= \frac{\Big\{-\Phi''V\cdot\Big(V^3-\frac{3}{8}V'^2\Big)+2\Phi'V'\cdot\Big(V^3+\frac{3}{16}V'^2-\frac{3}{8}VV''\Big)\Big\}}{12\Phi''V^2} \\
h_7 &\rightarrow \frac{V\cdot 8\cdot 18^2\Phi''V\cdot 2\cdot 12V}{24\Phi''\cdot(18\cdot 8V)^2}=\frac{V}{8} \\
h_8 &\rightarrow \frac{32\cdot 18^2\Phi''V\cdot 2\cdot 12\cdot V'}{4\cdot 24\Phi''(18\cdot 8V)^2}=\frac{V'}{8V}\ .
\end{aligned}
\tag{25}
$$

Especially if we choose

$$V=-2\ , \tag{26}$$

we obtain,

$$
\begin{aligned}
&h_1\rightarrow\frac{1}{24}\ ,\quad h_2\rightarrow-\frac{1}{8}\ ,\quad h_3\rightarrow\frac{1}{2}\ ,\quad h_4\rightarrow-\frac{1}{6} \\
&h_5\rightarrow 0\ ,\quad h_6\rightarrow-\frac{1}{3}\ ,\quad h_7\rightarrow-\frac{1}{4}\ ,\quad h_8\rightarrow 0
\end{aligned}
\tag{27}
$$

and we find that the standard result (conformal anomaly of $\mathcal{N}=4$ super YM theory covariantly coupled with $\mathcal{N}=4$ conformal supergravity [25]) in (24) is reproduced [21, 26].

We should also note that the expression (22) cannot be rewritten as a sum of the Gauss-Bonnet invariant G and the square of the Weyl tensor F, which are given as

$$
\begin{aligned}
G &= R^2-4R_{ij}R^{ij}+R_{ijkl}R^{ijkl} \\
F &= \frac{1}{3}R^2-2R_{ij}R^{ij}+R_{ijkl}R^{ijkl}\ ,
\end{aligned}
\tag{28}
$$

This is the signal that the conformal symmetry is broken already in classical theory.

When ϕ is constant, only two terms corresponding to h_1 and h_2 survive in (22) :

$$\begin{aligned} T &= -\frac{1}{8\pi G}\left[h_1 R^2 + h_2 R_{ij}R^{ij}\right] \\ &= -\frac{1}{8\pi G}\left[\left(h_1 + \frac{1}{3}h_2\right)R^2 + \frac{1}{2}h_2\left(F - G\right)\right] . \end{aligned} \tag{29}$$

As h_1 depends on V, we may compare the result with the conformal anomaly from, say, scalar or spinor QED, or QCD in the phase where there are no background scalars and (or) spinors.. The structure of the conformal anomaly in such a theory has the following form

$$T = \hat{a}G + \hat{b}F + \hat{c}R^2 \ . \tag{30}$$

where

$$\hat{a} = \text{constant} + a_1 e^2 \ , \quad \hat{b} = \text{constant} + a_2 e^2 \ , \quad \hat{c} = a_3 e^2 \ . \tag{31}$$

Here e^2 is the electric charge (or g^2 in case of QCD). Imagine that one can identify e with the exponential of the constant dilaton (using holographic RG [27, 28]). a_1, a_2 and a_3 are some numbers. Comparing (29) and (30), we obtain

$$\hat{a} = -\hat{b} = \frac{h_2}{16\pi G} \ , \quad \hat{c} = -\frac{1}{8\pi G}\left(h_1 + \frac{1}{3}h_2\right) \ . \tag{32}$$

When Φ is small, one gets

$$\begin{aligned} h_1 &= \frac{1}{24}\left[1 - \frac{1}{8}\Phi + \frac{1}{8}\frac{\left(\Phi'\right)^2}{\Phi''}\right. \\ &\quad \left. + \frac{25}{2592}\Phi^2 - \frac{17}{216}\frac{\left(\Phi'\right)^2\Phi}{\Phi''} + \frac{1}{576}\frac{\left(\Phi'\right)^2}{V} + \frac{1}{96}\frac{\left(\Phi'\right)^4}{\left(\Phi''\right)^2} + \mathcal{O}\left(\Phi^3\right)\right] \\ h_2 &= -\frac{1}{8}\left[1 - \frac{1}{8}\Phi + \frac{1}{8}\frac{\left(\Phi'\right)^2}{\Phi''}\right. \\ &\quad \left. + \frac{5}{576}\Phi^2 - \frac{3}{64}\frac{\left(\Phi'\right)^2\Phi}{\Phi''} + \frac{1}{96}\frac{\left(\Phi'\right)^4}{\left(\Phi''\right)^2} + \mathcal{O}\left(\Phi^3\right)\right] \ . \end{aligned} \tag{33}$$

If one assumes

$$\Phi(\phi) = a\mathrm{e}^{b\phi} \ , \quad (|a| \ll 1) \ , \tag{34}$$

then

$$
\begin{aligned}
h_2 &= -\frac{1}{8}\left[1 - \frac{a^2}{36}\mathrm{e}^{2b\phi} + \mathcal{O}\left(a^3\right)\right] \\
h_1 + \frac{1}{3}h_2 &= \frac{a^2}{24}\left(-\frac{5}{162} + \frac{b^2}{576V}\right)\mathrm{e}^{2b\phi} + \mathcal{O}\left(a^3\right) \ . \qquad (35)
\end{aligned}
$$

Comparing (35) with (31) and (32) and assuming

$$
e^2 = \mathrm{e}^{2b\phi} \ , \qquad (36)
$$

we find

$$
\begin{aligned}
a_1 = -a_2 &= \frac{1}{16\pi G}\cdot\frac{1}{8}\cdot\frac{a^2}{36} \ , \\
a_3 &= -\frac{1}{8\pi G}\cdot\frac{a^2}{24}\cdot\left(-\frac{5}{162} + \frac{b^2}{576V}\right) \ . \qquad (37)
\end{aligned}
$$

Here V should be arbitrary but constant. We should note $\Phi(0) \neq 0$. One can absorb the difference into the redefinition of l since we need not to assume $\Phi(0) = 0$ in deriving the form of h_1 and h_2 in (23). Hence, this simple example suggests the way of comparison between SG side and QFT descriptions of non-conformal boundary theory.

In order that the region near the boundary at $\rho = 0$ is asymptotically AdS, we need to require $\Phi \to 0$ and $\Phi' \to 0$ when $\rho \to 0$. One can also confirm that $h_1 \to \frac{1}{24}$ and $h_2 \to -\frac{1}{8}$ in the limit of $\Phi \to 0$ and $\Phi' \to 0$ even if $\Phi'' \neq 0$ and $\Phi''' \neq 0$. In the AdS/CFT correspondence, h_1 and h_2 are related with the central charge c of the conformal field theory (or its analog for non-conformal theory). Since we have two functions h_1 and h_2, there are two ways to define the candidate c-function when the conformal field theory is deformed:

$$
c_1 = \frac{24\pi h_1}{G} \ , \quad c_2 = -\frac{8\pi h_2}{G} \ . \qquad (38)
$$

If we put $V(\phi) = 4\lambda^2 + \Phi(\phi)$, then $l = \left(\frac{12}{V(0)}\right)^{\frac{1}{2}}$. One should note that it is chosen $l = 1$ in (38). We can restore l by changing $h \to l^3 h$ and $k \to l^3 k$ and $\Phi' \to l\Phi'$, $\Phi'' \to l^2\Phi''$ and $\Phi''' \to l^3\Phi'''$ in (22). Then in the limit of $\Phi \to 0$, one gets

$$
c_1 \ , \quad c_2 \ \to \frac{\pi}{G}\left(\frac{12}{V(0)}\right)^{\frac{3}{2}} \ , \qquad (39)
$$

which agrees with the proposal of the previous work [29] in the limit. The c-function c_1 or c_2 in (38) is, of course, more general definition. It is interesting to study the behaviour of candidate c-function for explicit values of dilatonic potential at different limits. It also could be interesting to see what is the analogue of our dilaton-dependent c-function in non-commutative YM theory (without dilaton, see [30]).

3 Properties of c-function

The definitions of the c-functions in (13) and (38), are, however, not always good ones since our results are too wide. That is, we have obtained the conformal anomaly for arbitrary dilatonic background which may not be the solution of original $d = 5$ gauged supergravity. As only solutions of d5 gauged supergravity describe RG flows of dual QFT it is not strange that above candidate c-functions are not acceptable. They quickly become non-monotonic and even singular in explicit examples. They presumbly measure the deviations from SG description and should not be taken seriously. As pointed in [35], it might be necessary to impose the condition $\Phi' = 0$ on the conformal boundary. Such condition follows from the equations of motion of d5 gauged SG. Anyway as $\Phi' = 0$ on the boundary in the solution which has the asymptotic AdS region, we can add any function which proportional to the power of $\Phi' = 0$ to the previous expressions of the c-functions in (13) and (38). As a trial, if we put $\Phi' = 0$, we obtain

$$c = \frac{3}{2G}\left[\frac{l}{2} + \frac{1}{l}\frac{1}{\Phi(\phi_{(0)}) + \frac{2}{l^2}}\right] \tag{40}$$

instead of (13) and

$$\begin{aligned} c_1 &= \frac{2\pi}{3G}\frac{62208 + 22464\Phi + 2196\Phi^2 + 72\Phi^3 + \Phi^4}{(6+\Phi)^2(24+\Phi)(18+\Phi)} \\ c_2 &= \frac{3\pi}{G}\frac{288 + 72\Phi + \Phi^2}{(6+\Phi)^2(24+\Phi)} \end{aligned} \tag{41}$$

instead of (38). We should note that there disappear the higher derivative terms like Φ'' or Φ'''. That will be our final proposal for acceptable c-function in terms of dilatonic potential. The given c-functions in (41) also have the

property (39) and reproduce the known result for the central charge on the boundary. Since $\frac{d\Phi}{dz} \to 0$ in the asymptotically AdS region even if the region is UV or IR, the given c-functions in (40) and (41) have fixed points in the asymptotic AdS region $\frac{dc}{dU} = \frac{dc}{d\Phi}\frac{d\Phi}{d\phi}\frac{d\phi}{dU} \to 0$, where $U = \rho^{-\frac{1}{2}}$ is the radius coordinate in AdS or the energy scale of the boundary field theory.

We can now check the monotonity in the c-functions. For this purpose, we consider some examples. In [6] and [7], the following dilaton potentials appeared:

$$\begin{aligned} 4\lambda^2 + \Phi_{\mathrm{FGPW}}(\phi) &= 4\left(\exp\left[\left(\frac{4\phi}{\sqrt{6}}\right)\right] + 2\exp\left[-\left(\frac{2\phi}{\sqrt{6}}\right)\right]\right) && (42) \\ 4\lambda^2 + \Phi_{\mathrm{GPPZ}}(\phi) &= \frac{3}{2}\left(3 + \left(\cosh\left[\left(\frac{2\phi}{\sqrt{3}}\right)\right]\right)^2 + 4\cosh\left[\left(\frac{2\phi}{\sqrt{3}}\right)\right]\right) && (43) \end{aligned}$$

In both cases V is a constant and $V = -2$. In the classical solutions for the both cases, ϕ is the monotonically decreasing function of the energy scale $U = \rho^{-\frac{1}{2}}$ and $\phi = 0$ at the UV limit corresponding to the boundary. Then in order to know the energy scale dependences of c_1 and c_2, we only need to investigate the ϕ dependences of c_1 and c_2 in (41). As the potentials and also Φ have a minimum $\Phi = 0$ at $\phi = 0$, which corresponds to the UV boundary in the solutions in [6] and [7], and Φ is monotonicaly increasing function of the absolute value $|\phi|$, we only need to check the monotonities of c_1 and c_2 with respect to Φ when $\Phi \geq 0$. From (41), we find

$$\begin{aligned} &\frac{d(\ln c_1)}{d\Phi} \\ &= -\frac{18\,(622080 + 383616\Phi + 64296\Phi^2 + 4548\Phi^3 + 130\Phi^4 + \Phi^5)}{(6+\Phi)(18+\Phi)(24+\Phi)(62208 + 22464\Phi + 2196\Phi^2 + 72\Phi^3 + \Phi^4)} \\ &< 0 \\ &\frac{d(\ln c_2)}{d\Phi} = -\frac{5184 + 2304\Phi + 138\Phi^2 + \Phi^3}{(6+\Phi)(24+\Phi)(288 + 72\Phi + \Phi^2)} < 0\ . && (44) \end{aligned}$$

Therefore the c-functions c_1 and c_2 are monotonically decreasing functions of Φ or increasings function of the energy scale U as the c-function in [4, 7]. We should also note that the c-functions c_1 and c_2 are positive definite for non-negative Φ. For c in (40) for $d = 2$ case, it is very straightforward to check the monotonity and the positivity.

In [29], another c-function has been proposed in terms of the metric as follows:

$$c_{\mathrm{GPPZ}} = \left(\frac{dA}{dz}\right)^{-3} \ , \tag{45}$$

where the metric is given by

$$ds^2 = dz^2 + \mathrm{e}^{2A} dx_\mu dx^\mu \ . \tag{46}$$

The c-function (45) is positive and has a fixed point in the asymptotically AdS region again and the c-function is also monotonically increasing function of the energy scale. The c-functions (40) and (41) proposed in this paper are given in terms of the dilaton potential, not in terms of metric, but it might be interesting that the c-functions in (40) and (41) have the similar properties (positivity, monotonity and fixed point in the asymptotically AdS region). These properties could be understood from the equations of motion. When the metric has the form (46), the equations of motion are:

$$\phi'' + dA'\phi' = \frac{\partial \Phi}{\partial \phi} \ , \tag{47}$$

$$dA'' + d(A')^2 + \frac{1}{2}(\phi')^2 = -\frac{4\lambda^2 + \Phi}{d-1} \ , \tag{48}$$

$$A'' + d(A')^2 = -\frac{4\lambda^2 + \Phi}{d-1} \ . \tag{49}$$

Here $' \equiv \frac{d}{dz}$. From (47) and (48), we obtain

$$0 = 2(d-1)A'' + \phi'^2 \tag{50}$$

Then if $A'' = 0$, $\phi' = 0$, which tells that if $\frac{dc_{\mathrm{GPPZ}}}{dz} = 0$, then $\frac{dc_1}{dz} = \frac{dc_2}{dz} = 0$. Then c_{GPPZ} has a fixed point, c_1 and c_2 have a fixed point. From (47) and (48), we also obtain

$$0 = d(d-1)A'^2 + 4\lambda^2 + \Phi - \frac{1}{2}\phi'^2 \ . \tag{51}$$

Then at the fixed point where $\phi' = 0$, we obtain

$$0 = d(d-1)A'^2 + 4\lambda^2 + \Phi \ . \tag{52}$$

Therefore if $c_{\rm GPPZ}$ and A' is the monotonic function of z, V and also c_1 and c_2 are also monotonic function at least at the fixed point. We have to note that above considerations do not give the proof of equivalency of our proposal c-functions with other proposals. However, it is remarkable (at least, for a number of potentials) that they enjoy the similar properties: positivity, monotonity and existance of fixed points.

We can also consider other examples of c-function for different choices of dilatonic potential. In [31], several examples of the potentials in gauged supergravity are given. They appeared as a result of sphere reduction in M-theory or string theory, down to three or five dimensions. Their properties are described in detail in refs.[31]. The potentials have the following form:

$$4\lambda^2 + \Phi(\phi) = \frac{d(d-1)}{\frac{1}{a_1^2} - \frac{1}{a_1 a_2}} \left(\frac{1}{a_1^2} \mathrm{e}^{a_1 \phi} - \frac{1}{a_1 a_2} \mathrm{e}^{a_2 \phi} \right) . \tag{53}$$

Here a_1 and a_2 are constant parameters depending on the model. We also normalize the potential so that $4\lambda^2 + \Phi(\phi) \to d(d-1)$ when $\phi \to 0$. For simplicity, we choose $G = l = 1$ in this section.

For $\mathcal{N} = 1$ model in $D = d + 1 = 3$ dimensions

$$a_1 = 2\sqrt{2} , \quad a_2 = \sqrt{2} , \tag{54}$$

for $D = 3$, $\mathcal{N} = 2$, one gets

$$a_1 = \sqrt{6} , \quad a_2 = 2\sqrt{\frac{2}{3}} , \tag{55}$$

and for $D = 3$, $\mathcal{N} = 3$ model, we have

$$a_1 = \frac{4}{\sqrt{3}} , \quad a_2 = \sqrt{3} . \tag{56}$$

On the other hand, for $D = d + 1 = 5$, $\mathcal{N} = 1$ model, a_1 and a_2 are

$$a_1 = 2\sqrt{\frac{5}{3}} , \quad a_2 = \frac{4}{\sqrt{15}} . \tag{57}$$

The proposed c-functions have not acceptable behaviour for above potentials. (There seems to be no problem for 2d case.) The problem seems to be that

the solutions in above models have not asymptotic AdS region in UV but in IR. On the same time the conformal anomaly in (22) is evaluated as UV effect. If we assume that Φ in the expression of c-functions c_1 and c_2 vanishes at IR AdS region, Φ becomes negative. When Φ is negative, the properties of the c-functions c_1 and c_2 become bad, they are not monotonic nor positive, and furthermore they have a singularity in the region given by the solutions in [31]. Thus, for such type of potential other proposal for c-function which isnot related with conformal nomaly should be made.

Hence, we discussed the typical behaviour of candidate c-functions. However, it is not clear which role should play dilaton in above expressions as holographic RG coupling constant in dual QFT. It could be induced mass, quantum fields or coupling constants (most probably, gauge coupling), but the explicit rule with what it should be identified is absent. The big number of usual RG parameters in dual QFT suggests also that there should be considered gauged SG with few scalars.

4 Surface Counterterms and Finite Action

As well-known, we need to add the surface terms to the bulk action in order to have the well-defined variational principle. Under the variation of the metric $\hat{G}^{\mu\nu}$ and the scalar field ϕ, the variation of the action (1) is given by

$$
\begin{aligned}
&\delta S = \delta S_{M_{d+1}} + \delta S_{M_d} \qquad (58)\\
&\delta S_{M_{d+1}} = \frac{1}{16\pi G}\int_{M_{d+1}} d^{d+1}x\sqrt{-\hat{G}}\left[\delta\hat{G}^{\zeta\xi}\left\{-\frac{1}{2}G_{\zeta\xi}\left\{\hat{R}\right.\right.\right.\\
&\quad \left. + \left(X(\phi)-Y'(\phi)\right)(\hat{\nabla}\phi)^2 + \Phi(\phi) + 4\lambda^2\right\} + \hat{R}_{\zeta\xi} + \left(X(\phi)-Y'(\phi)\right)\partial_\zeta\phi\partial_\xi\phi\Big\}\\
&\quad +\delta\phi\left\{\left(X'(\phi)-Y''(\phi)\right)(\hat{\nabla}\phi)^2 + \Phi'(\phi)\right.\\
&\quad \left.\left. -\frac{1}{\sqrt{-\hat{G}}}\partial_\mu\left(\sqrt{-\hat{G}}\hat{G}^{\mu\nu}\left(X(\phi)-Y'(\phi)\right)\partial_\nu\phi\right)\right\}\right] .\\
&\delta S_{M_d} = \frac{1}{16\pi G}\int_{M_d} d^dx\sqrt{-\hat{g}}n_\mu\left[\partial^\mu\left(\hat{G}_{\xi\nu}\delta\hat{G}^{\xi\nu}\right) - D_\nu\left(\delta\hat{G}^{\mu\nu}\right) + Y(\phi)\partial^\mu\left(\delta\phi\right)\right] .
\end{aligned}
$$

Here $\hat{g}_{\mu\nu}$ is the metric induced from $\hat{G}_{\mu\nu}$ and n_μ is the unit vector normal to M_d. The surface term δS_{M_d} of the variation contains $n^\mu\partial_\mu\left(\delta\hat{G}^{\xi\nu}\right)$ and

$n^\mu\partial_\mu\left(\delta\phi\right)$, which makes the variational principle ill-defined. In order that the variational principle is well-defined on the boundary, the variation of the action should be written as

$$\delta S_{M_d} = \lim_{\rho\to 0}\int_{M_d} d^dx\sqrt{-\hat g}\left[\delta\hat G^{\xi\nu}\left\{\cdots\right\} + \delta\phi\left\{\cdots\right\}\right] \tag{59}$$

after using the partial integration. If we put $\{\cdots\}=0$ for $\{\cdots\}$ in (59), one could obtain the boundary condition corresponding to Neumann boundary condition. We can, of course, select Dirichlet boundary condition by choosing $\delta\hat G^{\xi\nu} = \delta\phi = 0$, which is natural for AdS/CFT correspondence. The Neumann type condition becomes, however, necessary later when we consider the black hole mass etc. by using surface terms. If the variation of the action on the boundary contains $n^\mu\partial_\mu\left(\delta\hat G^{\xi\nu}\right)$ or $n^\mu\partial_\mu\left(\delta\phi\right)$, however, we cannot partially integrate it on the boundary in order to rewrite the variation in the form of (59) since n_μ expresses the direction perpendicular to the boundary. Therefore the "minimum" of the action is ambiguous. Such a problem was well studied in [10] for the Einstein gravity and the boundary term was added to the action. It cancels the term containing $n^\mu\partial_\mu\left(\delta\hat G^{\xi\nu}\right)$. We need to cancel also the term containing $n^\mu\partial_\mu\left(\delta\phi\right)$. Then one finds the boundary term [21]

$$S_b^{(1)} = -\frac{1}{8\pi G}\int_{M_d} d^dx\sqrt{-\hat g}\left[D_\mu n^\mu + Y(\phi)n_\mu\partial^\mu\phi\right] \ . \tag{60}$$

We also need to add surface counterterm $S_b^{(2)}$ which cancels the divergence coming from the infinite volume of the bulk space, say AdS. In order to investigate the divergence, we choose the metric in the form (5). In the parametrization (5), n^μ and the curvature R are given by

$$\begin{aligned} n^\mu &= \left(\frac{2\rho}{l}, 0, \cdots, 0\right) \\ R &= \tilde R + \frac{3\rho^2}{l^2}\hat g^{ij}\hat g^{kl}\hat g'_{ik}\hat g'_{jl} - \frac{4\rho^2}{l^2}\hat g^{ij}\hat g''_{ij} - \frac{\rho^2}{l^2}\hat g^{ij}\hat g^{kl}\hat g'_{ij}\hat g'_{kl} \ . \end{aligned} \tag{61}$$

Here $\tilde R$ is the scalar curvature defined by g_{ij} in (5). Expanding g_{ij} and ϕ with respect to ρ as in (7), we find the following expression for $S+S_b^{(1)}$:

$$S + S_b^{(1)} = \frac{1}{16\pi G}\lim_{\rho\to 0}\int d^dx l\rho^{-\frac{d}{2}}\sqrt{-g_{(0)}}\left[\frac{2-2d}{l^2} - \frac{1}{d}\Phi(\phi_0)\right.$$

$$+\rho\left\{-\frac{1}{d-2}R_{(0)}-\frac{1}{l^2}g_{(0)}^{ij}g_{(1)ij}\right.$$
$$-\frac{1}{d-2}\Bigg(X(\phi_{(0)})\left(\nabla_{(0)}\phi_{(0)}\right)^2+Y(\phi_{(0)})\Delta\phi_{(0)}$$
$$\left.\left.+\Phi'(\phi_{(0)})\phi_{(1)}\Bigg)\right\}+\mathcal{O}\left(\rho^2\right)\right] \ . \tag{62}$$

Then for $d=2$

$$S_b^{(2)}=\frac{1}{16\pi G}\int d^dx\sqrt{-\hat{g}}\left[\frac{2}{l}+\frac{l}{2}\Phi(\phi)\right] \tag{63}$$

and for $d=3,4$,

$$S_b^{(2)}=\frac{1}{16\pi G}\int d^dx\left[\sqrt{-\hat{g}}\left\{\frac{2d-2}{l}+\frac{l}{d-2}R-\frac{2l}{d(d-2)}\Phi(\phi)\right.\right.$$
$$\left.\left.+\frac{l}{d-2}\left(X(\phi)\left(\hat{\nabla}\phi\right)^2+Y(\phi)\hat{\Delta}\phi\right)\right\}-\frac{l^2}{d(d-2)}n^\mu\partial_\mu\left(\sqrt{-\hat{g}}\Phi(\phi)\right)\right] \tag{64}$$

Note that the last term in above expression does not look typical from the AdS/CFT point of view. The reason is that it does not depend from only the boundary values of the fields. Its presence may indicate to breaking of AdS/CFT conjecture in the situations when SUGRA scalars significally deviate from constants or are not asymptotic constants [5].

Here $\hat{\Delta}$ and $\hat{\nabla}$ are defined by using d-dimensional metric and we used

$$\sqrt{-\hat{g}}\Phi(\phi)=\rho^{-\frac{d}{2}}\sqrt{-g_{(0)}}\Bigg\{\Phi(\phi_{(0)})$$
$$+\rho\left(\frac{1}{2}g_{(0)}^{ij}g_{(1)ij}\Phi(\phi_{(0)})+\Phi'(\phi_{(0)})\phi_{(1)}\right)+\mathcal{O}\left(\rho^2\right)\Bigg\}$$
$$n^\mu\partial_\mu\left(\sqrt{-\hat{g}}\Phi(\phi)\right)=\frac{2}{l}\rho^{-\frac{d}{2}}\sqrt{-g_{(0)}}\Bigg\{-\frac{d}{2}\Phi(\phi_{(0)}$$
$$+\rho\left(1-\frac{d}{2}\right)\left(\frac{1}{2}g_{(0)}^{ij}g_{(1)ij}\Phi(\phi_{(0)})+\Phi'(\phi_{(0)})\phi_{(1)}\right)+\mathcal{O}\left(\rho^2\right)\Bigg\} \ . \tag{65}$$

Note that $S_b^{(2)}$ in (63) or (64) is only given in terms of the boundary quantities except the last term in (64). The last term is necessary to cancel the

[5] We thank the referee for adressing this issue.

divergence of the bulk action and it is, of course, the total derivative in the bulk theory:

$$\int d^d x n^\mu \partial_\mu \left(\sqrt{-\hat{g}}\Phi(\phi)\right) = \int d^{d+1}x\sqrt{-\hat{G}}\Box\Phi(\phi) \ . \tag{66}$$

Thus we got the boundary counterterm action for gauged SG. Using these local surface counterterms as part of complete action one can show explicitly that bosonic sector of gauged SG in dimensions under discussion gives finite action in asymptotically AdS space. The corresponding example will be given in next section.

Recently the surface counterterms for the action with the dilaton (scalar) potential are discussed in [19]. Their counterterms seem to correspond to the terms cancelling the leading divergence when $\rho \to 0$ in (62). However, they seem to have only considered the case where the dilaton becomes asymptotically constant $\phi \to \phi_0$. If we choose $\phi_0 = 0$, the total dilaton potential including the cosmological term $V_{\rm dilaton}(\phi) \equiv 4\lambda^2 + \Phi(\phi)$ approaches to $V_{\rm dilaton}(\phi) \to 4\lambda^2 = d(d-1)/l^2$. Then if we only consider the leading ρ behavior and the asymptotically constant dilaton, the counterterm action in (63) and/or (64) has the following form

$$S_b^{(2)} = \frac{1}{16\pi G}\int d^d x\sqrt{-\hat{g}}\left(\frac{2d-2}{l}\right) \ , \tag{67}$$

which coincides with the result in [19] when the spacetime is aymptotically AdS.

Let us turn now to the discussion of deep connection between surface counterterms and holographic conformal anomaly. It is enough to mention only $d = 4$. In order to control the logarithmically divergent terms in the bulk action S, we choose $d - 4 = \epsilon < 0$. Then

$$S + S_b = \frac{1}{\epsilon}S_{\rm ln} + \text{ finite terms} \ . \tag{68}$$

Here $S_{\rm ln}$ is given in (14). We also find

$$g_{(0)}^{ij}\frac{\delta}{\delta g_{(0)}^{ij}}S_{\rm ln} = -\frac{\epsilon}{2}\mathcal{L}_{\rm ln} + \mathcal{O}\left(\epsilon^2\right) \ . \tag{69}$$

Here $\mathcal{L}_{\rm ln}$ is the Lagrangian density corresponding to $S_{\rm ln}$: $S_{\rm ln} = \int d^{d+1}\mathcal{L}_{\rm ln}$. Then combining (68) and (69), we obtain the trace anomaly :

$$T = \lim_{\epsilon\to 0-} \frac{2\hat{g}_{(0)}^{ij}}{\sqrt{-\hat{g}_{(0)}}} \frac{\delta(S+S_b)}{\delta \hat{g}_{(0)}^{ij}} = -\frac{1}{2}\mathcal{L}_{\rm ln} \ , \tag{70}$$

which is identical with the result found in (8). We should note that the last term in (64) does not lead to any ambiguity in the calculation of conformal anomaly since $g_{(0)}$ does not depend on ρ. If we use the equations of motion (15), (16), (17) and (18), we finally obtain the expression (22) or (111). Hence, we found the finite gravitational action (for asymptotically AdS spaces) in 5 dimensions by adding the local surface counterterm. This action correctly reproduces holographic trace anomaly for dual (gauge) theory. In principle, one can also generalize all results for higher dimensions, say, d6, etc. With the growth of dimension, the technical problems become more and more complicated as the number of structures in boundary term is increasing.

5 Dilatonic AdS Black Hole and its Mass

Let us consider the black hole or "throat" type solution for the equations of the motion (2) and (3) when $d = 4$. The surface term (64) may be used for calculation of the finite black hole mass and/or other thermodynamical quantities.

For simplicity, we choose

$$X(\phi) = \alpha \ (\text{constant}) \ , \quad Y(\phi) = 0 \tag{71}$$

and we assume the spacetime metric in the following form:

$$ds^2 = -\mathrm{e}^{2\rho}dt^2 + \mathrm{e}^{2\sigma}dr^2 + r^2\sum_{i=1}^{d-1}\left(dx^i\right)^2 \tag{72}$$

and ρ, σ and ϕ depend only on r. The equations (2) and (3) can be rewritten in the following form:

$$0 \ = \ \mathrm{e}^{\rho+\sigma}\Phi'(\phi) - 2\alpha\left(\mathrm{e}^{\rho-\sigma}\phi'\right)' \tag{73}$$

$$0 = -\frac{1}{3}\mathrm{e}^{2\rho}\left(\Phi(\phi)+\frac{12}{l^2}\right)+\left(\rho''+(\rho')^2-\rho'\sigma'+\frac{3\rho'}{r}\right)\mathrm{e}^{2\rho-2\sigma} \tag{74}$$

$$0 = \frac{1}{3}\mathrm{e}^{2\sigma}\left(\Phi(\phi)+\frac{12}{l^2}\right)-\rho''-(\rho')^2+\rho'\sigma'+\frac{3\sigma'}{r}+\alpha\,(\phi')^2 \tag{75}$$

$$0 = \frac{1}{3}\mathrm{e}^{2\sigma}\left(\Phi(\phi)+\frac{12}{l^2}\right)r^2+k+\{r\,(\sigma'-\rho')-2\}\,\mathrm{e}^{-2\sigma}\ . \tag{76}$$

Here $' \equiv \frac{d}{dr}$. If one defines new variables U and V by

$$U=\mathrm{e}^{\rho+\sigma}\ ,\quad V=r^2\mathrm{e}^{\rho-\sigma}\ , \tag{77}$$

we obtain the following equations from (73-76):

$$0 = r^3U\Phi'(\phi)-2\alpha\,(rV\phi')' \tag{78}$$

$$0 = \frac{1}{3}\mathrm{e}^{2\sigma}\left(\Phi(\phi)+\frac{12}{l^2}\right)r^3U+kr-V' \tag{79}$$

$$0 = \frac{3U'}{rU}+\alpha\,(\phi)'\ . \tag{80}$$

We should note that only three equations in (73-76) are independent. There is practical problem in the construction of AdS BH with non-trivial dilaton, especially for arbitrary dilatonic potential. That is why we use below the approximate technique which was developed in ref.[32] for constant dilatonic potential.

When $\Phi(0)=\Phi'(0)=\phi=0$, a solution corresponding to the throat limit of D3-brane is given by

$$U=1\ ,\quad V=V_0\equiv\frac{r^4}{l^2}-\mu\ . \tag{81}$$

In the following, we use large r expansion and consider the perturbation around (81). It is assumed

$$\Phi(\phi)=\tilde\mu\phi^2+\mathcal{O}\left(\phi^3\right)\ . \tag{82}$$

Then one can neglect the higher order terms in (82). We obtain from (78)

$$0\sim\tilde\mu r^3\phi+\alpha\left(\frac{r^5}{l^2}\phi'\right)'\ . \tag{83}$$

The solution of eq.(83) is given by

$$\phi = cr^{-\beta} \ , \ (c \text{ is a constant}) \ , \beta = 2 \pm \sqrt{4 - \frac{\tilde{\mu} l^2}{\alpha}} \ . \tag{84}$$

Consider r is large or c is small, and write U and V in the following form:

$$U = 1 + c^2 u \ , \quad V = V_0 + c^2 v \ . \tag{85}$$

Then from (79) and(80), one gets

$$u = u_0 + \frac{\alpha\beta}{6} r^{-2\beta} \ , \quad v = v_0 - \frac{\tilde{\mu}(\beta - 6)}{6(\beta - 4)(\beta - 2)} r^{-2\beta+4} \ . \tag{86}$$

Here u_0 and v_0 are constants of the integration. Here we choose

$$v_0 = u_0 = 0 \ . \tag{87}$$

The horizon which is defined by

$$V = 0 \tag{88}$$

lies at

$$r = r_h \equiv l^{\frac{1}{2}} \mu^{\frac{1}{4}} + c^2 \frac{\tilde{\mu}(\beta - 6) l^{\frac{5}{2} - \beta} \mu^{\frac{1}{4} - \frac{\beta}{2}}}{24(\beta - 4)(\beta - 2)} \ . \tag{89}$$

And the Hawking temperature is

$$\begin{aligned} T &= \frac{1}{4\pi} \left[\frac{1}{r^2} \frac{dV}{dr} \right]_{r=r_h} \\ &= \frac{1}{4\pi} \left\{ 4 l^{-\frac{3}{2}} \mu^{\frac{1}{4}} + c^2 \frac{\tilde{\mu}(\beta - 6)(2\beta - 3)}{6(\beta - 4)(\beta - 2)} l^{\frac{1}{2} - \beta} \mu^{\frac{1}{4} - \frac{\beta}{2}} \right\} \ . \end{aligned} \tag{90}$$

We now evaluate the free energy of the black hole within the standard prescription [33, 34]. The free energy F can be obtained by substituting the classical solution into the action S:

$$F = TS \ . \tag{91}$$

Here T is the Hawking temperature. Using the equations of motion in (2) ($X = \alpha$, $Y = 0$, $4\lambda^2 = \frac{12}{l^2}$), we obtain

$$0 = \frac{5}{3}\left(\Phi(\phi) + \frac{12}{l^2}\right) + \hat{R} + \alpha\left(\nabla\phi\right)^2 \ . \tag{92}$$

Substituting (92) into the action (1) after Wick-rotating it to the Euclid signature

$$\begin{aligned} S &= \frac{1}{16\pi G}\cdot\frac{2}{3}\int_{M_5} d^5\sqrt{G}\left(\Phi(\phi) + \frac{12}{l^2}\right) \\ &= \frac{1}{16\pi G}\cdot\frac{2}{3}\frac{V_{(3)}}{T}\int_{r_h}^{\infty} drr^3 U\left(\Phi(\phi) + \frac{12}{l^2}\right) \ . \end{aligned} \tag{93}$$

Here $V_{(3)}$ is the volume of the 3d space ($\int d^5x \cdots = \beta V_{(3)} \int drr^3 \cdots$) and β is the period of time, which can be regarded as the inverse of the temperature T ($\frac{1}{T}$). The expression (93) contains the divergence. We regularize the divergence by replacing

$$\int^{\infty} dr \to \int^{r_{\max}} dr \tag{94}$$

and subtract the contribution from a zero temperature solution, where we choose $\mu = c = 0$, and the solution corresponds to the vacuum or pure AdS:

$$S_0 = \frac{1}{16\pi G}\cdot\frac{2}{3}\cdot\frac{12}{l^2}\frac{V_{(3)}}{T}\sqrt{\frac{G_{tt}\left(r = r_{\max}, \mu = c = 0\right)}{G_{tt}\left(r = r_{\max}\right)}}\int_{r_h}^{\infty} drr^3 \ . \tag{95}$$

The factor $\sqrt{\frac{G_{tt}(r=r_{\max},\mu=c=0)}{G_{tt}(r=r_{\max})}}$ is chosen so that the proper length of the circles which correspond to the period $\frac{1}{T}$ in the Euclid time at $r_{\max}$ coincides with each other in the two solutions. Then we find the following expression for the free energy,

$$\begin{aligned} F &= \lim_{r_{\max}\to\infty} T\left(S - S_0\right) \\ &= \frac{V_{(3)}}{2\pi G l^2 T^2}\left[-\frac{l^2\mu}{8} + c^2\mu^{1-\frac{\beta}{2}}\tilde{\mu}\left\{\frac{(\beta-1)}{12\beta(\beta-4)(\beta-2)}\right\} + \cdots\right] \ . \end{aligned} \tag{96}$$

Here we assume $\beta > 2$ or the expression $S - S_0$ still contains the divergences and we cannot get finite results. However, the inequality $\beta > 2$ is not always

satisfied in the gauged supergravity models. In that case the expression in (96) would not be valid. One can express the free energy F in (96) in terms of the temperature T instead of μ:

$$F = \frac{V_{(3)}}{16\pi G}\left[-\pi T^4 l^6 + c^2 l^{8-4\beta} T^{4-2\beta}\tilde{\mu}\left(\frac{2\beta^3 - 15\beta^2 + 22\beta - 4}{6\beta(\beta-4)(\beta-2)}\right) + \cdots\right] . \tag{97}$$

Then the entropy $\mathcal{S}$ and the energy (mass) E is given by

$$\begin{aligned}
\mathcal{S} &= -\frac{dF}{dT} = \frac{V_{(3)}}{16\pi G}\Big[4\pi T^3 l^6 \\
&\quad + c^2 l^{8-4\beta} T^{3-2\beta}\tilde{\mu}\left(\frac{2\beta^3 - 15\beta^2 + 22\beta - 4}{3\beta(\beta-4)}\right) + \cdots\Big] \\
E &= F + TS = \frac{V_{(3)}}{16\pi G}\Big[3\pi T^4 l^6 \\
&\quad + c^2 l^{8-4\beta}\left(\pi T^4\right)^{1-\frac{\beta}{2}}\tilde{\mu}\left(\frac{(2\beta-3)(2\beta^3 - 15\beta^2 + 22\beta - 4)}{6\beta(\beta-4)(\beta-2)}\right) + \cdots\Big]
\end{aligned} \tag{98}$$

We now evaluate the mass using the surface term of the action in (64), i.e. within local surface counterterm method. The surface energy momentum tensor T_{ij} is now defined by $(d = 4)$[6]

$$\begin{aligned}
\delta S_b^{(2)} &= \sqrt{-\hat{g}}\,\delta\hat{g}^{ij} T_{ij} \\
&= \frac{1}{16\pi G}\left[\sqrt{-\hat{g}}\,\delta\hat{g}^{ij}\left\{-\frac{1}{2}\hat{g}_{ij}\left(\frac{6}{l} + \frac{l}{2}\hat{R} + \frac{l}{4}\Phi(\phi)\right)\right\}\right. \\
&\quad \left. + \frac{l^2}{4} n^\mu \partial_\mu\left\{\sqrt{-\hat{g}}\,\delta\hat{g}^{ij}\hat{g}_{ij}\Phi(\phi)\right\}\right] .
\end{aligned} \tag{99}$$

[6] S does not contribute due to the equation of motion in the bulk. The variation of $S + S_b^{(1)}$ gives a contribution proportional to the extrinsic curvature θ_{ij} at the boundary:

$$\delta\left(S + S_b^{(2)}\right) = \frac{\sqrt{-\hat{g}}}{16\pi G}\left(\theta_{ij} - \theta\hat{g}_{ij}\right)\delta\hat{g}^{ij}$$

The contribution is finite even in the limit of $r \to \infty$. Then the finite part does not depend on the parameters characterizing the black hole. Therefore after subtracting the contribution from the reference metric, which could be that of AdS, the contribution from the variation of $S + S_b^{(1)}$ vanishes.

Note that the energy-momentum tensor is still not well-defined due to the term containing $n^\mu \partial_\mu$. If we assume $\delta\hat{g}^{ij} \sim \mathcal{O}\left(\rho^{a_1}\right)$ for large ρ when we choose the coordinate system (5), then

$$n^\mu \partial_\mu \left(\delta\hat{g}^{ij}\cdot\right) \sim \frac{2}{l}\delta\hat{g}^{ij}\left(a_1 + \partial_\rho\right)(\cdot) \ . \tag{100}$$

Or if $\delta\hat{g}^{ij} \sim \mathcal{O}\left(r^{a_2}\right)$ for large r when we choose the coordinate system (72), then

$$n^\mu \partial_\mu \left(\delta\hat{g}^{ij}\cdot\right) \sim \delta\hat{g}^{ij}\mathrm{e}^\sigma \left(\frac{a_2}{r} + \partial_r\right)(\cdot) \ . \tag{101}$$

As we consider the black hole-like object in this section, one chooses the coordinate system (72) and assumes Eq.(101). Then mass E of the black hole like object is given by

$$E = \int d^{d-1}x\sqrt{\tilde{\sigma}}N\delta T_{tt}\left(u^t\right)^2 \ . \tag{102}$$

Here we assume the metric of the reference spacetime (e.g. AdS) has the form of $ds^2 = f(r)dr^2 - N^2(r)dt^2 + \sum_{i,j=1}^{d-1}\tilde{\sigma}_{ij}dx^i dx^j$ and δT_{tt} is the difference of the (t,t) component of the energy-momentum tensor in the spacetime with black hole like object from that in the reference spacetime, which we choose to be AdS, and u^t is the t component of the unit time-like vector normal to the hypersurface given by t =constant. By using the solution in (85) and (86), the (t,t) component of the energy-momentum tensor in (99) has the following form:

$$\begin{aligned} T_{tt} &= \frac{3}{16\pi G}\frac{r^2}{l^3}\left[1 - \frac{l^3\mu}{r^4} + l^2\tilde{\mu}c^2\left(\frac{1}{12} - \frac{1}{6\beta(\beta-6)}\right.\right. \\ &\qquad \left.\left. - \frac{\beta-6}{6(\beta-4)(\beta-2)} - \frac{(3-\beta)(1+a_2)}{12}\right)r^{-2\beta} + \cdots\right] \ . \end{aligned} \tag{103}$$

If we assume the mass is finite, β should satisfy the inequality $\beta > 2$, as in the case of the free energy in (96) since $\sqrt{\sigma}N\left(u^t\right)^2 = lr^2$ for the reference AdS space. Then the β-dependent term in (103) does not contribute to the mass and one gets

$$E = \frac{3\mu V_{(3)}}{16\pi G} \ . \tag{104}$$

Using (90)

$$E = \frac{3l^6 V_{(3)} \pi T^4}{16\pi G} \left\{ 1 - c^2 \tilde{\mu} l^{2-4\beta} \left(\pi T^4\right)^{-\frac{\beta}{2}} \frac{(\beta-6)(2\beta-3)}{(\beta-4)(\beta-2)} \right\} \ , \tag{105}$$

which does not agree with the result in (98). This might express the ambiguity in the choice of the regularization to make the finite action. A possible origin of it might be following. We assumed ϕ can be expanded in the (integer) power series of ρ in (7) when deriving the surface terms in (64). However, this assumption seems to conflict with the classical solution in (84), where the fractional power seems to appear since $r^2 \sim \frac{1}{\rho}$. In any case, in QFT there is no problem in regularization dependence of the results. In many cases (see example in ref.[17]) the explicit choice of free parameters of regularization leads to coincidence of the answers which look different in different regularizations. As usually happens in QFT the renormalization is more universal as the same answers for beta-functions may be obtained while using different regularizations. That suggests that holographic renormalization group should be developed and the predictions of above calculations should be tested in it.

As in the case of the c-function, we might be drop the terms containing Φ' in the expression of $S_b^{(2)}$ in (64). Then we obtain

$$\begin{aligned} S_b^{(2)} = \frac{1}{16\pi G} \int d^d x \Bigg[\sqrt{-\hat{g}} \Bigg\{ \frac{2d-2}{l} + \frac{l}{d-2} R + \frac{2l}{d(d-2)} \Phi(\phi) \\ + \frac{l}{d-2} \left(X(\phi) \left(\hat{\nabla}\phi\right)^2 + Y(\phi) \hat{\Delta}\phi \right) \Bigg\} - \frac{l^2 \Phi(\phi)}{d(d-2)} n^\mu \partial_\mu \left(\sqrt{-\hat{g}} \right) \Bigg] . \end{aligned} \tag{106}$$

If we use the expression (106), however, the result of the mass E in (105) does not change.

6 Discussion

In summary, we constructed surface counterterm for gauged supergravity with single scalar and arbitrary scalar potential in three and five dimensions. As a result, the finite gravitational action and consistent stress tensor in asymptotically AdS space is found. Using this action, the regularized expressions for free energy, entropy and mass are derived for d5 dilatonic

AdS black hole. From another side, finite action may be used to get the holographic conformal anomaly of boundary QFT with broken conformal invariance. Such conformal anomaly is calculated from d5 and d3 gauged SG with arbitrary dilatonic potential with the use of AdS/CFT correspondence. Due to dilaton dependence it takes extremely complicated form. Within holographic RG where identification of dilaton with some coupling constant is made, we suggested the candidate c-function for d2 and d4 boundary QFT from holographic conformal anomaly. It is shown that such proposal gives monotonic and positive c-function for few examples of dilatonic potential.

We expect that our results may be very useful in explicit identification of supergravity description (special RG flow) with the particular boundary gauge theory (or its phase) which is very non-trivial task in AdS/CFT correspondence. We show that on the example of constant dilaton and special form of dilatonic potential where qualitative agreement of holographic conformal anomaly and QFT conformal anomaly (with the account of radiative corrections) from QED-like theory with single coupling constant may be achieved.

Our work may be extended in various directions. First of all, we can consider large number of scalars, say 42 as in $N = 8$ d5 SG, and construct the corresponding Weyl anomaly from the bulk side. However, this is technically very complicated problem as even in case of single scalar the complete answer for d4 anomaly takes few pages. The calculation of surface counterterm in d5 gauged SG with many scalars is slightly easier task. However, again the application of surface counterterm for the derivation of regularized thermodynamical quantities in multi-scalar AdS black holes (when they will be constructed) is complicated. Second, the generalization of surface counterterm for higher dimensions (say, $d = 7, 9$) is possible. Third, in general the extension of AdS/CFT set-up to non-conformal boundary theories is challenging problem. In this respect, better investigation of candidate c-functions from bulk and from boundary is required as well as their comparison in all detail. The related question is bulk calculation of Casimir effect in the presence of dilaton and comparison of it with QFT result, including radiative corrections.

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A Coefficients of conformal anomaly

In this appendix, we give the explicit values of the coefficients appeared in the calculation of $d=4$ conformal anomaly.

Substituting (16) into (15), we obtain

$$\begin{aligned}
g_{(1)ij} &= \tilde{c}_1 R_{ij} + \tilde{c}_2 g_{ij} R + \tilde{c}_3 g_{ij} g^{kl} \partial_k\phi\partial_l\phi \\
&\quad + \tilde{c}_4 g_{ij} \frac{\partial_k}{\sqrt{-g}} \left(\sqrt{-g} g^{kl}\partial_l\phi\right) + \tilde{c}_5 \partial_i\phi\partial_j\phi \qquad (107)
\end{aligned}$$

$$\begin{aligned}
\tilde{c}_1 &= -\frac{3}{6+\Phi} \\
\tilde{c}_2 &= -\frac{3\ \{\Phi'^2 - 6\ (\Phi'' + 8\ V)\}}{2\ (6+\Phi)\ \{-2\ \Phi'^2 + (18+\Phi)\ (\Phi'' + 8\ V)\}} \\
\tilde{c}_3 &= \frac{-3\ \Phi'^2\ V + 18\ V\ (\Phi'' + 8\ V) - 2\ (6+\Phi)\ \Phi'\ V'}{2\ (6+\Phi)\ (-2\ \Phi'^2 + (18+\Phi)\ (\Phi'' + 8\ V))} \\
\tilde{c}_4 &= -\frac{2\ \Phi'\ V}{-2\ \Phi'^2 + (18+\Phi)\ (\Phi'' + 8\ V)} \\
\tilde{c}_5 &= -\frac{V}{2+\frac{\Phi}{3}}\ . \qquad (108)
\end{aligned}$$

Further, substituting (16) and (107) into (18), we obtain

$$\begin{aligned}
\phi_{(2)} &= d_1 R^2 + d_2 R_{ij}R^{ij} + d_3 R^{ij}\partial_i\phi\partial_j\phi \\
&\quad + d_4 R g^{ij}\partial_i\phi\partial_j\phi + d_5 R \frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}g^{ij}\partial_j\phi) \\
&\quad + d_6 (g^{ij}\partial_i\phi\partial_j\phi)^2 + d_7 \left(\frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}g^{ij}\partial_j\phi)\right)^2 \\
&\quad + d_8 g^{kl}\partial_k\phi\partial_l\phi \frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}g^{ij}\partial_j\phi) \qquad (109)
\end{aligned}$$

$$d_1 = -\Big[9\ \Phi'\ \Big\{2\ (12+\Phi)\ \Phi'^4 - (\,-864 + 36\ \Phi + 24\ \Phi^2 + \Phi^3)\ \Phi''^2$$

$$
\begin{aligned}
& +192\ (12+\Phi)^2\ \Phi''\ V + 64\ (2592 + 612\ \Phi + 48\ \Phi^2 + \Phi^3)\ V^2 \\
& -2\ \Phi'^2\ \Big((216 + 30\ \Phi + \Phi^2)\ \Phi'' + 144\ (10+\Phi)\ V\Big) \\
& +(6+\Phi)^2\ (24+\Phi)\ \Phi'\ (\Phi''' + 8\ V')\Big\}\Big]\ / \\
& \Big[8\ (6+\Phi)^2\ \Big\{-2\ \Phi'^2 + (24+\Phi)\ \Phi''\Big\} \\
& \times\Big\{-2\ \Phi'^2 + (18+\Phi)\ (\Phi'' + 8\ V)\Big\}^2\Big] \\
d_2 &= \frac{9\ (12+\Phi)\ \Phi'}{4\ (6+\Phi)^2\ \{-2\ \Phi'^2 + (24+\Phi)\ \Phi''\}} \\
d_3 &= \frac{3\ (3\ (12+\Phi)\ \Phi'\ V - 2\ (144 + 30\ \Phi + \Phi^2)\ V')}{2\ (6+\Phi)^2\ (-2\ \Phi'^2 + (24+\Phi)\ \Phi'')} \\
d_4 &= (3\ (-6\ (12+\Phi)\ \Phi'^5\ V + 6\ (108 + 24\ \Phi + \Phi^2)\ \Phi'^4\ V' \\
& +4\ (2592 + 684\ \Phi + 48\ \Phi^2 + \Phi^3)\ (\Phi'' + 8\ V)\ ((9+\Phi)\ \Phi'' \\
& +4\ (12+\Phi)\ V)\ V' - (6+\Phi)\ \Phi'^2\ (3\ (144 + 30\ \Phi + \Phi^2)\ \Phi'''\ V \\
& +(1980\ \Phi'' + 216\ \Phi\ \Phi'' + 5\ \Phi^2\ \Phi'' + 27360\ V + 4176\ \Phi\ V \\
& +128\ \Phi^2\ V)\ V') + 2\ \Phi'^3\ (3\ (216 + 30\ \Phi + \Phi^2)\ \Phi''\ V \\
& -2\ (-2160\ V^2 - 216\ \Phi\ V^2 + 864\ V'' + 324\ \Phi\ V'' \\
& +36\ \Phi^2\ V'' + \Phi^3\ V'')) + \Phi'\ (3\ (-864 + 36\ \Phi + 24\ \Phi^2 + \Phi^3)\ \Phi''^2\ V \\
& +2\ \Phi''\ (-41472\ V^2 - 6912\ \Phi\ V^2 - 288\ \Phi^2\ V^2 + 15552\ V'' \\
& +6696\ \Phi\ V'' + 972\ \Phi^2\ V'' + 54\ \Phi^3\ V'' + \Phi^4\ V'') \\
& -2\ (248832\ V^3 + 58752\ \Phi\ V^3 + 4608\ \Phi^2\ V^3 \\
& +96\ \Phi^3\ V^3 + 15552\ \Phi'''\ V' + 6696\ \Phi\ \Phi'''\ V' + 972\ \Phi^2\ \Phi'''\ V' \\
& +54\ \Phi^3\ \Phi'''\ V' + \Phi^4\ \Phi'''\ V' + 124416\ V'^2 + 53568\ \Phi\ V'^2 \\
& +7776\ \Phi^2\ V'^2 + 432\ \Phi^3\ V'^2 + 8\ \Phi^4\ V'^2 - 124416\ V\ V'' \\
& -53568\ \Phi\ V\ V'' - 7776\ \Phi^2\ V\ V'' - 432\ \Phi^3\ V\ V'' \\
& -8\ \Phi^4\ V\ V''))))/ \\
& (4\ (6+\Phi)^2\ (-2\ \Phi'^2 + (24+\Phi)\ \Phi'')\ (-2\ \Phi'^2 \\
& +(18+\Phi)\ (\Phi'' + 8\ V))^2) \\
d_5 &= -(3\ (2\ \Phi'^4\ V + 2\ (432 + 42\ \Phi + \Phi^2)\ \Phi''\ V\ (\Phi'' + 8\ V) \\
& +\Phi'^2\ V\ ((6+\Phi)\ \Phi'' - 8\ (162 + 7\ \Phi)\ V) - 4\ (24+\Phi)\ \Phi'^3\ V' \\
& -2\ (432 + 42\ \Phi + \Phi^2)\ \Phi'\ (\Phi'''\ V - \Phi''\ V')))/
\end{aligned}
$$

$$(2\ (2\ \Phi'^2 - (24+\Phi)\ \Phi'')\ (-2\ \Phi'^2 + (18+\Phi)\ (\Phi'' + 8\ V))^2)$$

$$\begin{aligned}
d_6 \;=\; & -(-54\ (12+\Phi)\ \Phi'^5\ V^2 + 12\ (828 + 168\ \Phi + 5\ \Phi^2)\ \Phi'^4\ V\ V' \\
& +4\ (2592 + 684\ \Phi + 48\ \Phi^2 + \Phi^3) \\
& V'\ (54\ \Phi''^2\ V + 4608\ V^3 + 192\ \Phi\ V^3 + 108\ \Phi'''\ V' + 24\ \Phi\ \Phi'''\ V' \\
& +\Phi^2\ \Phi'''\ V' + 864\ V'^2 + 192\ \Phi\ V'^2 + 8\ \Phi^2\ V'^2 - 1728\ V\ V'' \\
& -384\ \Phi\ V\ V'' - 16\ \Phi^2\ V\ V'' + 2\ \Phi''\ (504\ V^2 + 12\ \Phi\ V^2 - 108\ V'' \\
& -24\ \Phi\ V'' - \Phi^2\ V'')) + (6+\Phi)\ \Phi'^2\ (9\ (144 + 30\ \Phi + \Phi^2)\ \Phi'''\ V^2 \\
& -2\ V'\ (14796\ \Phi''\ V + 1368\ \Phi\ \Phi''\ V + 33\ \Phi^2\ \Phi''\ V \\
& +88992\ V^2 + 4680\ \Phi\ V^2 + 36\ \Phi^2\ V^2 - 20736\ V'' \\
& -5472\ \Phi\ V'' - 384\ \Phi^2\ V'' - 8\ \Phi^3\ V'')) \\
& +2\ \Phi'^3\ (27\ (216 + 30\ \Phi + \Phi^2)\ \Phi''\ V^2 + 4\ (12312\ V^3 \\
& +1836\ \Phi\ V^3 + 72\ \Phi^2\ V^3 + 2376\ V'^2 + 864\ \Phi\ V'^2 + 90\ \Phi^2\ V'^2 \\
& +2\ \Phi^3\ V'^2 + 2592\ V\ V'' + 972\ \Phi\ V\ V'' + 108\ \Phi^2\ V\ V'' \\
& +3\ \Phi^3\ V\ V'')) - \Phi'\ (27\ (2304 + 516\ \Phi + 40\ \Phi^2 + \Phi^3)\ \Phi''^2\ V^2 \\
& +4\ \Phi''\ (217728\ V^3 + 44064\ \Phi\ V^3 + 3024\ \Phi^2\ V^3 + 72\ \Phi^3\ V^3 \\
& +81648\ V'^2 + 34992\ \Phi\ V'^2 + 5040\ \Phi^2\ V'^2 + 276\ \Phi^3\ V'^2 \\
& +5\ \Phi^4\ V'^2 + 46656\ V\ V'' + 20088\ \Phi\ V\ V'' \\
& +2916\ \Phi^2\ V\ V'' + 162\ \Phi^3\ V\ V'' + 3\ \Phi^4\ V\ V'') \\
& +4\ V\ (746496\ V^3 + 129600\ \Phi\ V^3 + 6912\ \Phi^2\ V^3 \\
& +144\ \Phi^3\ V^3 - 46656\ \Phi'''\ V' - 20088\ \Phi\ \Phi'''\ V' - 2916\ \Phi^2\ \Phi'''\ V' \\
& -162\ \Phi^3\ \Phi'''\ V' - 3\ \Phi^4\ \Phi'''\ V' - 404352\ V'^2 - 177984\ \Phi\ V'^2 \\
& -26784\ \Phi^2\ V'^2 - 1584\ \Phi^3\ V'^2 - 32\ \Phi^4\ V'^2 + 373248\ V\ V'' \\
& +160704\ \Phi\ V\ V'' + 23328\ \Phi^2\ V\ V'' + 1296\ \Phi^3\ V\ V'' \\
& +24\ \Phi^4\ V\ V'')))/(8\ (6+\Phi)^2\ (-2\ \Phi'^2 + (24+\Phi)\ \Phi'')\ (-2\ \Phi'^2 \\
& +(18+\Phi)\ (\Phi'' + 8\ V))^2) \\
d_7 \;=\; & (2\ V\ (36\ \Phi'^3\ V - 3\ (18+\Phi)\ \Phi'\ V \\
& ((26+\Phi)\ \Phi'' - 8\ (18+\Phi)\ V) + 4\ (432 + 42\ \Phi + \Phi^2)\ \Phi'^2\ V' \\
& +(18+\Phi)^2\ (24+\Phi)\ (\Phi'''\ V - 2\ (\Phi'' + 4\ V)\ V')))/ \\
& ((2\ \Phi'^2 - (24+\Phi)\ \Phi'')\ (-2\ \Phi'^2 + (18+\Phi)\ (\Phi'' + 8\ V))^2) \\
d_8 \;=\; & -(6\ \Phi'^4\ V^2 - 4\ (156 + 5\ \Phi)\ \Phi'^3\ V\ V' - 2\ (18+\Phi)\ \Phi'\ V
\end{aligned}$$

$$
\begin{aligned}
&(3\ (24+\Phi)\ \Phi'''\ V + (-276\ \Phi'' - 11\ \Phi\ \Phi'' + 480\ V + 32\ \Phi\ V)\ V') \\
&+2\ (432 + 42\ \Phi + \Phi^2)\ (3\ \Phi''^2\ V^2 \\
&+2\ (18+\Phi)\ V\ (-\Phi'''\ V' + 8\ V\ V'') \\
&+2\ \Phi''\ (12\ V^3 + 18\ V'^2 + \Phi\ V'^2 + 18\ V\ V'' + \Phi\ V\ V'')) \\
&+\Phi'^2\ (3\ (6+\Phi)\ \Phi''\ V^2 - 8\ (486\ V^3 + 21\ \Phi\ V^3 + 432\ V'^2 \\
&+42\ \Phi\ V'^2 + \Phi^2\ V'^2 + 432\ V\ V'' + 42\ \Phi\ V\ V'' + \Phi^2\ V\ V'')))/ \\
&(2\ (2\ \Phi'^2 - (24+\Phi)\ \Phi'')\ \left(-2\ \Phi'^2 + (18+\Phi)\ (\Phi'' + 8\ V)\right)^2)\ .
\end{aligned}
$$

Substituting (16), (107) and (109) into (17), one gets

$$
\begin{aligned}
g^{ij}g_{(2)ij} &= f_1 R^2 + f_2 R_{ij}R^{ij} + f_3 R^{ij}\partial_i\phi\partial_j\phi \\
&\quad + f_4 R g^{ij}\partial_i\phi\partial_j\phi + f_5 R\frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}g^{ij}\partial_j\phi) \\
&\quad + f_6 (g^{ij}\partial_i\phi\partial_j\phi)^2 + f_7\left(\frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}g^{ij}\partial_j\phi)\right)^2 \\
&\quad + f_8 g^{kl}\partial_k\phi\partial_l\phi\frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}g^{ij}\partial_j\phi) \qquad (110) \\
f_1 &= \Big[9\ \Big\{2\ \Phi'^6 - 72\ (12+\Phi)\ \Phi''\ (\Phi'' + 8\ V)^2 \\
&\quad -2\ \Phi'^4\ ((24+\Phi)\ \Phi'' + 8\ (18+\Phi)\ V) \\
&\quad +\Phi'^2\ \Big((324 + 12\ \Phi - \Phi^2)\ \Phi''^2 \\
&\quad +8\ (540 + 48\ \Phi + \Phi^2)\ \Phi''\ V + 64\ (180 + 24\ \Phi + \Phi^2)\ V^2\Big) \\
&\quad +(6+\Phi)^2\ \Phi'^3\ (\Phi''' + 8\ V')\Big\}\Big]\ / \\
&\quad \Big[2\ (6+\Phi)^2\ \Big\{-2\ \Phi'^2 + (24+\Phi)\ \Phi''\Big\} \\
&\quad \times\Big\{-2\ \Phi'^2 + (18+\Phi)\ (\Phi'' + 8\ V)\Big\}^2\Big] \\
f_2 &= -\frac{9\ (\Phi'^2 - 6\ \Phi'')}{(6+\Phi)^2\ \{-2\ \Phi'^2 + (24+\Phi)\ \Phi''\}} \\
f_3 &= \frac{6\ (-3\ \Phi'^2\ V + 18\ \Phi''\ V + 2\ (6+\Phi)\ \Phi'\ V')}{(6+\Phi)^2\ (-2\ \Phi'^2 + (24+\Phi)\ \Phi'')} \\
f_4 &= -(3\ (-12\ \Phi'^6\ V + 432\ (12+\Phi)\ \Phi''\ V\ (\Phi'' + 8\ V)^2 \\
&\quad +8\ (6+\Phi)\ \Phi'^5\ V' + (6+\Phi)\ \Phi'\ ((1044 + 168\ \Phi + 7\ \Phi^2)\ \Phi''^2
\end{aligned}
$$

$$
\begin{aligned}
& +8\ (1476+192\ \Phi+7\ \Phi^2)\ \Phi''\ V \\
& +256\ (216+30\ \Phi+\Phi^2)\ V^2)\ V' \\
& -2\ (6+\Phi)\ \Phi'^3\ (3\ (6+\Phi)\ \Phi'''\ V \\
& +(66\ \Phi''+3\ \Phi\ \Phi''+912\ V+88\ \Phi\ V)\ V') \\
& +4\ \Phi'^4\ (3\ (24+\Phi)\ \Phi''\ V-2\ (-216\ V^2-12\ \Phi\ V^2+36\ V'' \\
& +12\ \Phi\ V''+\Phi^2\ V''))+2\ \Phi'^2\ (3\ (-324-12\ \Phi+\Phi^2)\ \Phi''^2\ V \\
& +2\ (18+\Phi)\ \Phi''\ (-360\ V^2-12\ \Phi\ V^2+36\ V''+12\ \Phi\ V''+\Phi^2\ V'') \\
& -2\ (17280\ V^3+2304\ \Phi\ V^3 \\
& +96\ \Phi^2\ V^3+648\ \Phi'''\ V'+252\ \Phi\ \Phi'''\ V'+30\ \Phi^2\ \Phi'''\ V' \\
& +\Phi^3\ \Phi'''\ V'+5184\ V'^2+2016\ \Phi\ V'^2+240\ \Phi^2\ V'^2+8\ \Phi^3\ V'^2 \\
& -5184\ V\ V''-2016\ \Phi\ V\ V''-240\ \Phi^2\ V\ V''-8\ \Phi^3\ V\ V''))))/ \\
& (2\ (6+\Phi)^2\ (-2\ \Phi'^2+(24+\Phi)\ \Phi'')\ (-2\ \Phi'^2+(18+\Phi)\ (\Phi''+8\ V))^2) \\
f_5 \;=\;& -(3\ \Phi'\ (\Phi''\ V\ (-3\ (10+\Phi)\ \Phi''+8\ (42+\Phi)\ V) \\
& +\Phi'^2\ (-6\ \Phi''\ V+32\ V^2)+8\ \Phi'^3\ V' \\
& +4\ (18+\Phi)\ \Phi'\ (\Phi'''\ V-\Phi''\ V')))/ \\
& ((2\ \Phi'^2-(24+\Phi)\ \Phi'')\ (-2\ \Phi'^2+(18+\Phi)\ (\Phi''+8\ V))^2) \\
f_6 \;=\;& (-54\ \Phi'^6\ V^2+72\ (6+\Phi)\ \Phi'^5\ V\ V' \\
& +2\ \Phi''(54\ (252+30\ \Phi+\Phi^2)\ \Phi''^2\ V^2 \\
& +24\ V\ (36288\ V^3+4320\ \Phi\ V^3+144\ \Phi^2\ V^3 \\
& +11664\ V'^2+5184\ \Phi\ V'^2+792\ \Phi^2\ V'^2+48\ \Phi^3\ V'^2+\Phi^4\ V'^2) \\
& +\Phi''\ (217728\ V^3+25920\ \Phi\ V^3+864\ \Phi^2\ V^3 \\
& +11664\ V'^2+5184\ \Phi\ V'^2+792\ \Phi^2\ V'^2+48\ \Phi^3\ V'^2+\Phi^4\ V'^2)) \\
& +(6+\Phi)\ \Phi'^3\ (9\ (6+\Phi)\ \Phi'''\ V^2-2\ V'\ (666\ \Phi''\ V+39\ \Phi\ \Phi''\ V \\
& +4392\ V^2+156\ \Phi\ V^2-864\ V''-192\ \Phi\ V''-8\ \Phi^2\ V'')) \\
& +(6+\Phi)\ \Phi'\ V'\ (3\ (1548+120\ \Phi+\Phi^2)\ \Phi''^2\ V \\
& +8\ \Phi''\ (11124\ V^2+1152\ \Phi\ V^2 \\
& +27\ \Phi^2\ V^2-1944\ V''-540\ \Phi\ V'' \\
& -42\ \Phi^2\ V''-\Phi^3\ V'')+4\ (18+\Phi) \\
& (4608\ V^3+192\ \Phi\ V^3+108\ \Phi'''\ V' \\
& +24\ \Phi\ \Phi'''\ V'+\Phi^2\ \Phi'''\ V'+864\ V'^2
\end{aligned}
$$

$$
\begin{aligned}
&+192\ \Phi\ V'^2+8\ \Phi^2\ V'^2-1728\ V\ V''-384\ \Phi\ V\ V''-16\ \Phi^2\ V\ V''))\\
&+6\ \Phi'^4\ (9\ (24+\Phi)\ \Phi''\ V^2+4\ (324\ V^3+18\ \Phi\ V^3\\
&+36\ V'^2+12\ \Phi\ V'^2+\Phi^2\ V'^2+36\ V\ V''+12\ \Phi\ V\ V''+\Phi^2\ V\ V''))\\
&-\Phi'^2\ (27\ (396+36\ \Phi+\Phi^2)\ \Phi''^2\ V^2+4\ \Phi''\ (29160\ V^3\\
&+2592\ \Phi\ V^3+54\ \Phi^2\ V^3+4104\ V'^2+1620\ \Phi\ V'^2+198\ \Phi^2\ V'^2\\
&+7\ \Phi^3\ V'^2+1944\ V\ V''+756\ \Phi\ V\ V''+90\ \Phi^2\ V\ V''+3\ \Phi^3\ V\ V'')\\
&+4\ V\ (67392\ V^3+6912\ \Phi\ V^3+144\ \Phi^2\ V^3\\
&-1944\ \Phi'''\ V'-756\ \Phi\ \Phi'''\ V'\\
&-90\ \Phi^2\ \Phi'''\ V'-3\ \Phi^3\ \Phi'''\ V'-5184\ V'^2-2016\ \Phi\ V'^2-240\ \Phi^2\ V'^2\\
&-8\ \Phi^3\ V'^2+15552\ V\ V''+6048\ \Phi\ V\ V''\\
&+720\ \Phi^2\ V\ V''+24\ \Phi^3\ V\ V'')))/\\
&(2\ (6+\Phi)^2\ (-2\ \Phi'^2+(24+\Phi)\ \Phi'')\ (-2\ \Phi'^2\\
&+(18+\Phi)\ (\Phi''+8\ V))^2)
\end{aligned}
$$

$$
\begin{aligned}
f_7 \;=\;& -(4\ V\ (4\ \Phi'^4\ V-2\ (78+5\ \Phi)\ \Phi'^2\ \Phi''\ V\\
&+(18+\Phi)^2\ \Phi''\ V\ (\Phi''+24\ V)\\
&+8\ (18+\Phi)\ \Phi'^3\ V'+2\ (18+\Phi)^2\ \Phi'\ (\Phi'''\ V-2\ (\Phi''+4\ V)\ V')))/\\
&((2\ \Phi'^2-(24+\Phi)\ \Phi'')\ (-2\ \Phi'^2+(18+\Phi)\ (\Phi''+8\ V))^2)\\
f_8 \;=\;& (-56\ \Phi'^4\ V\ V'-4\ (18+\Phi)^2\ \Phi''\ V\ (\Phi''+24\ V)\ V'\\
&-4\ \Phi'^2\ V\ (3\ (18+\Phi)\ \Phi'''\ V\\
&-(246\ \Phi''+15\ \Phi\ \Phi''+288\ V+16\ \Phi\ V)\ V')\\
&+2\ \Phi'^3\ (9\ \Phi''\ V^2-8\ (6\ V^3+18\ V'^2+\Phi\ V'^2\\
&+18\ V\ V''+\Phi\ V\ V''))+\Phi'\\
&(9\ (10+\Phi)\ \Phi''^2\ V^2\\
&+8\ (18+\Phi)^2\ V\ (-\Phi'''\ V'+8\ V\ V'')-8\ \Phi''\ (126\ V^3+3\ \Phi\ V^3\\
&-324\ V'^2-36\ \Phi\ V'^2-\Phi^2\ V'^2\\
&-324\ V\ V''-36\ \Phi\ V\ V''-\Phi^2\ V\ V'')))/\\
&((2\ \Phi'^2-(24+\Phi)\ \Phi'')\ (-2\ \Phi'^2+(18+\Phi)\ (\Phi''+8\ V))^2)\ .
\end{aligned}
$$

Finally substituting (16), (107), (109) and (110) into the expression for the anomaly (14), we obtain,

$$
T \;=\; -\frac{1}{8\pi G}\left[h_1R^2+h_2R_{ij}R^{ij}+h_3R^{ij}\partial_i\phi\partial_j\phi\right.
$$

$$
\begin{aligned}
&+h_4 R g^{ij}\partial_i\phi\partial_j\phi + h_5 R\frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}g^{ij}\partial_j\phi)\\
&+h_6(g^{ij}\partial_i\phi\partial_j\phi)^2 + h_7\left(\frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}g^{ij}\partial_j\phi)\right)^2\\
&\left.+h_8 g^{kl}\partial_k\phi\partial_l\phi\frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}g^{ij}\partial_j\phi)\right]
\end{aligned}
\tag{111}
$$

$$
\begin{aligned}
h_1 \;=\;& \Big[3\ \Big\{(24-10\ \Phi)\ \Phi'^6\\
&+(62208+22464\ \Phi+2196\ \Phi^2+72\ \Phi^3+\Phi^4)\ \Phi''\ (\Phi''+8\ V)^2\\
&+2\ \Phi'^4\ \Big\{(108+162\ \Phi+7\ \Phi^2)\ \Phi''+72\ (-8+14\ \Phi+\Phi^2)\ V\Big\}\\
&-2\ \Phi'^2\ \Big\{(6912+2736\ \Phi+192\ \Phi^2+\Phi^3)\ \Phi''^2\\
&+4\ (11232+6156\ \Phi+552\ \Phi^2+13\ \Phi^3)\ \Phi''\ V\\
&+32\ (-2592+468\ \Phi+96\ \Phi^2+5\ \Phi^3)\ V^2\Big\}\\
&-3\ (-24+\Phi)\ (6+\Phi)^2\ \Phi'^3\ (\Phi'''+8\ V')\Big\}\Big]\ /\\
&\Big[16\ (6+\Phi)^2\ \Big\{-2\ \Phi'^2+(24+\Phi)\ \Phi''\Big\}\ \Big\{-2\ \Phi'^2\\
&+(18+\Phi)\ (\Phi''+8\ V)\Big\}^2\Big]\\
h_2 \;=\;& -\frac{3\ \{(12-5\ \Phi)\ \Phi'^2+(288+72\ \Phi+\Phi^2)\ \Phi''\}}{8\ (6+\Phi)^2\ \{-2\ \Phi'^2+(24+\Phi)\ \Phi''\}}\\
h_3 \;=\;& -(3\ ((12-5\ \Phi)\ \Phi'^2\ V+(288+72\ \Phi+\Phi^2)\ \Phi''\ V\\
&+2\ (-144-18\ \Phi+\Phi^2)\ \Phi'\ V'))/\\
&(4\ (6+\Phi)^2\ (-2\ \Phi'^2+(24+\Phi)\ \Phi''))\\
h_4 \;=\;& (-6\ (-12+5\ \Phi)\ \Phi'^6\ V\\
&+3\ (62208+22464\ \Phi+2196\ \Phi^2+72\ \Phi^3+\Phi^4)\ \Phi''\ V\ (\Phi''+8\ V)^2\\
&+2\ (-684-48\ \Phi+11\ \Phi^2)\ \Phi'^5\ V'\\
&+(6+\Phi)\ \Phi'\ ((-31104-2772\ \Phi+120\ \Phi^2+13\ \Phi^3)\ \Phi''^2\\
&+8\ (-62208-7092\ \Phi-132\ \Phi^2+7\ \Phi^3)\ \Phi''\ V\\
&+384\ (-5184-504\ \Phi+6\ \Phi^2+\Phi^3)\ V^2)\ V'\\
&-(6+\Phi)\ \Phi'^3\ (9\ (-144-18\ \Phi+\Phi^2)\ \Phi'''\ V\\
&+(-3492\ \Phi''+252\ \Phi\ \Phi''+19\ \Phi^2\ \Phi''\\
&-71712\ V-4944\ \Phi\ V+208\ \Phi^2\ V)\ V')
\end{aligned}
$$

$$
\begin{aligned}
& +6\ \Phi'^4\ ((108+162\ \Phi+7\ \Phi^2)\ \Phi''\ V+2\ (\,-288\ V^2 \\
& +504\ \Phi\ V^2+36\ \Phi^2\ V^2+864\ V''+252\ \Phi\ V''+12\ \Phi^2\ V''-\Phi^3\ V'')) \\
& -6\ \Phi'^2\ ((6912+2736\ \Phi+192\ \Phi^2+\Phi^3)\ \Phi''^2\ V \\
& -82944\ V^3+14976\ \Phi\ V^3 \\
& +3072\ \Phi^2\ V^3+160\ \Phi^3\ V^3-15552\ \Phi'''\ V' \\
& -5400\ \Phi\ \Phi'''\ V'-468\ \Phi^2\ \Phi'''\ V' \\
& +6\ \Phi^3\ \Phi'''\ V'+\Phi^4\ \Phi'''\ V'-124416\ V'^2 \\
& -43200\ \Phi\ V'^2-3744\ \Phi^2\ V'^2 \\
& +48\ \Phi^3\ V'^2+8\ \Phi^4\ V'^2+124416\ V\ V'' \\
& +43200\ \Phi\ V\ V''+3744\ \Phi^2\ V\ V'' \\
& -48\ \Phi^3\ V\ V''-8\ \Phi^4\ V\ V'' \\
& +\Phi''\ (44928\ V^2+24624\ \Phi\ V^2+2208\ \Phi^2\ V^2 \\
& +52\ \Phi^3\ V^2+15552\ V''+5400\ \Phi\ V'' \\
& +468\ \Phi^2\ V''-6\ \Phi^3\ V''-\Phi^4\ V'')))/ \\
& (8\ (6+\Phi)^2\ (\,-2\ \Phi'^2+(24+\Phi)\ \Phi'')\ (\,-2\ \Phi'^2 \\
& +(18+\Phi)\ (\Phi''+8\ V))^2) \\
h_5 \;=\; & (\Phi'\ (\,-10\ \Phi'^4\ V+\Phi'^2\ V\ ((426+\Phi)\ \Phi''-8\ (270+\Phi)\ V) \\
& +\Phi\ \Phi''\ V\ (-7\ (6+\Phi)\ \Phi''+8\ (174+5\ \Phi)\ V)+12\ (-24+\Phi)\ \Phi'^3\ V' \\
& +6\ (\,-432-6\ \Phi+\Phi^2)\ \Phi'\ (\Phi'''\ V-\Phi''\ V')))/ \\
& (4\ (2\ \Phi'^2-(24+\Phi)\ \Phi'')\ (\,-2\ \Phi'^2+(18+\Phi)\ (\Phi''+8\ V))^2) \\
h_6 \;=\; & (18\ (-12+5\ \Phi)\ \Phi'^6\ V^2+4\ (2772+384\ \Phi-13\ \Phi^2)\ \Phi'^5\ V\ V' \\
& -\Phi''\ (3\ (124416+44928\ \Phi \\
& +4212\ \Phi^2+144\ \Phi^3+\Phi^4)\ \Phi''^2\ V^2 \\
& +48\ \Phi''\ (124416\ V^3+44928\ \Phi\ V^3+4212\ \Phi^2\ V^3+144\ \Phi^3\ V^3+\Phi^4\ V^3 \\
& -23328\ V'^2-10368\ \Phi\ V'^2-1584\ \Phi^2\ V'^2-96\ \Phi^3\ V'^2-2\ \Phi^4\ V'^2) \\
& +64\ V\ (373248\ V^3+134784\ \Phi\ V^3 \\
& +12636\ \Phi^2\ V^3+432\ \Phi^3\ V^3+3\ \Phi^4\ V^3-139968\ V'^2 \\
& -50544\ \Phi\ V'^2-4320\ \Phi^2\ V'^2+216\ \Phi^3\ V'^2+36\ \Phi^4\ V'^2+\Phi^5\ V'^2)) \\
& -(6+\Phi)\ \Phi'^3\ (9\ (\,-144-18\ \Phi+\Phi^2)\ \Phi'''\ V^2 \\
& -2\ V'\ (\,-17244\ \Phi''\ V-540\ \Phi\ \Phi''\ V+29\ \Phi^2\ \Phi''\ V-99360\ V^2
\end{aligned}
$$

$$
\begin{aligned}
&+1992\ \Phi\ V^2+212\ \Phi^2\ V^2+20736\ V''+3744\ \Phi\ V''-8\ \Phi^3\ V''))\\
&+2\ (6+\Phi)\ \Phi'\ V'\ ((62208+3708\ \Phi-24\ \Phi^2+\Phi^3)\ \Phi''^2\ V\\
&-4\ \Phi''\ (-248832\ V^2-11736\ \Phi\ V^2+840\ \Phi^2\ V^2+34\ \Phi^3\ V^2\\
&+46656\ V''+11016\ \Phi\ V''+468\ \Phi^2\ V''-18\ \Phi^3\ V''-\Phi^4\ V'')\\
&-2\ (-432-6\ \Phi+\Phi^2)\ (4608\ V^3+192\ \Phi\ V^3\\
&+108\ \Phi'''\ V'+24\ \Phi\ \Phi'''\ V'+\Phi^2\ \Phi'''\ V'+864\ V'^2\\
&+192\ \Phi\ V'^2+8\ \Phi^2\ V'^2-1728\ V\ V''\\
&-384\ \Phi\ V\ V''-16\ \Phi^2\ V\ V''))\\
&-2\ \Phi'^4\ (3\ (180+438\ \Phi+17\ \Phi^2)\ \Phi''\ V^2+4\ (-4752\ V^3\\
&+1116\ \Phi\ V^3+66\ \Phi^2\ V^3-3240\ V'^2-1008\ \Phi\ V'^2-66\ \Phi^2\ V'^2\\
&+2\ \Phi^3\ V'^2-2592\ V\ V''-756\ \Phi\ V\ V''-36\ \Phi^2\ V\ V''+3\ \Phi^3\ V\ V''))\\
&+4\ \Phi'^2\ (6\ (2484+1197\ \Phi+84\ \Phi^2+2\ \Phi^3)\ \Phi''^2\ V^2\\
&+\Phi''\ (88128\ V^3+67608\ \Phi\ V^3+5040\ \Phi^2\ V^3+90\ \Phi^3\ V^3-125712\ V'^2\\
&-46656\ \Phi\ V'^2-4896\ \Phi^2\ V'^2-72\ \Phi^3\ V'^2+5\ \Phi^4\ V'^2-46656\ V\ V''\\
&-16200\ \Phi\ V\ V''-1404\ \Phi^2\ V\ V''+18\ \Phi^3\ V\ V''+3\ \Phi^4\ V\ V'')\\
&+3\ V\ (-82944\ V^3+30528\ \Phi\ V^3\\
&+3840\ \Phi^2\ V^3+80\ \Phi^3\ V^3+15552\ \Phi'''\ V'\\
&+5400\ \Phi\ \Phi'''\ V'+468\ \Phi^2\ \Phi'''\ V'-6\ \Phi^3\ \Phi'''\ V'-\Phi^4\ \Phi'''\ V'+72576\ V'^2\\
&+28224\ \Phi\ V'^2+3360\ \Phi^2\ V'^2+112\ \Phi^3\ V'^2-124416\ V\ V''\\
&-43200\ \Phi\ V\ V''-3744\ \Phi^2\ V\ V''+48\ \Phi^3\ V\ V''+8\ \Phi^4\ V\ V'')))/\\
&(16\ (6+\Phi)^2\ (-2\ \Phi'^2+(24+\Phi)\ \Phi'')\ (-2\ \Phi'^2\\
&+(18+\Phi)\ (\Phi''+8\ V))^2)\\
h_7\ =\ &-(V\ (84\ \Phi'^4\ V-8\ (18+\Phi)^2\ \Phi''\ V\ (-3\ \Phi''+2\ (-12+\Phi)\ V)\\
&+\Phi'^2\ V\ (3\ (-876-40\ \Phi+\Phi^2)\ \Phi''+8\ (18+\Phi)^2\ V)\\
&-4\ (-432-6\ \Phi+\Phi^2)\ \Phi'^3\ V'\\
&-(-24+\Phi)\ (18+\Phi)^2\ \Phi'\ (\Phi'''\ V-2\ (\Phi''+4\ V)\ V')))/\\
&((2\ \Phi'^2-(24+\Phi)\ \Phi'')\ (-2\ \Phi'^2+(18+\Phi)\ (\Phi''+8\ V))^2)\\
h_8\ =\ &(-10\ \Phi'^5\ V^2+4\ (-204+5\ \Phi)\ \Phi'^4\ V\ V'\\
&+32\ (18+\Phi)^2\ \Phi''\ V\ (-3\ \Phi''+2\ (-12+\Phi)\ V)\ V'\\
&+2\ \Phi'^2\ V\ (3\ (-432-6\ \Phi+\Phi^2)\ \Phi'''\ V
\end{aligned}
$$

$$
\begin{aligned}
&+(7416\ \Phi'' + 270\ \Phi\ \Phi'' - 11\ \Phi^2\ \Phi'' \\
&+1728\ V - 480\ \Phi\ V - 32\ \Phi^2\ V)\ V') \\
&+\Phi'^3\ ((426+\Phi)\ \Phi''\ V^2 - 8\ (270\ V^3 + \Phi\ V^3 \\
&+432\ V'^2 + 6\ \Phi\ V'^2 - \Phi^2\ V'^2 + 432\ V\ V'' + 6\ \Phi\ V\ V'' - \Phi^2\ V\ V'')) \\
&+\Phi'\ (-6\ \Phi\ (7\ \Phi''^2\ V^2 - 232\ \Phi''\ V^3 + 360\ \Phi'''\ V\ V' \\
&-360\ \Phi''\ V'^2 - 360\ \Phi''\ V\ V'' - 2880\ V^2\ V'') \\
&+4\ \Phi^3\ (\Phi'''\ V\ V' - \Phi''\ V'^2 - \Phi''\ V\ V'' - 8\ V^2\ V'') \\
&+31104\ (-\Phi'''\ V\ V' + \Phi''\ V'^2 + \Phi''\ V\ V'' + 8\ V^2\ V'') \\
&-\Phi^2\ (7\ \Phi''^2\ V^2 - 40\ \Phi''\ V^3 - 48\ \Phi'''\ V\ V' + 48\ \Phi''\ V'^2 \\
&+48\ \Phi''\ V\ V'' + 384\ V^2\ V'')))/ \\
&(4\ (2\ \Phi'^2 - (24+\Phi)\ \Phi'')\ (-2\ \Phi'^2 + (18+\Phi)\ (\Phi'' + 8\ V))^2)\ .
\end{aligned}
$$

The c functions proposed in this paper for $d = 4$ case is given by h_1 and h_2 by putting Φ' to vanish:

$$
\begin{aligned}
c_1 &= \frac{2\pi}{3G}\frac{62208 + 22464\Phi + 2196\Phi^2 + 72\Phi^3 + \Phi^4}{(6+\Phi)^2(24+\Phi)(18+\Phi)} \\
c_2 &= \frac{3\pi}{G}\frac{288 + 72\Phi + \Phi^2}{(6+\Phi)^2(24+\Phi)}
\end{aligned}
\tag{112}
$$

Note also that using of above condition on the zero value of dilatonic potential derivative on conformal boundary significally simplifies the conformal anomaly as many terms vanish.

B Comparison with other counterterm schemes

In this Appendix, we compare the counter terms and the trace anomaly obtained here with those in ref.[36] which appeared after this work has been submitted to hepth. For simplicity, we consider the case that the spacetime dimension is 4 and the boundary is flat and the metric g_{ij} in (5) on the boundary is given by

$$
g_{ij} = F(\rho)\eta_{ij}\ . \tag{113}
$$

We also assume the dilaton ϕ only depends on ρ: $\phi = \phi(\rho)$. This is exactly the case of ref.[36]. Then the conformal anomaly (22) vanishes on such background.

Let us demonstrate that this is consistent with results of ref.[36]. In the metric (113), the equation of motion (2) given by the variation over the dilaton ϕ and the Einstein equations in (3) have the following forms:

$$0 = -\frac{l}{2\rho^3}F^2\Phi'(\phi) - \frac{2}{l}\partial_\rho\left(\frac{F^2}{\rho}\partial_\rho\phi\right) \tag{114}$$

$$0 = \frac{l^2}{12\rho^2}\left(\Phi(\phi) + \frac{12}{l^2}\right) - \frac{1}{\rho^2} - \frac{2}{F}\partial_\rho^2 F + \frac{1}{F^2}\left(\partial_\rho F\right)^2 - \frac{1}{2}\left(\partial_\rho\phi\right)^2 \tag{115}$$

$$0 = \frac{F}{3\rho}\left(\Phi(\phi) + \frac{12}{l^2}\right) - \frac{2\rho}{l^2}\partial_\rho^2 F - \frac{2\rho}{l^2 F}\left(\partial_\rho F\right)^2 + \frac{6}{l^2}\partial_\rho F - \frac{4F}{l^2\rho} . \tag{116}$$

Eq.(115) corresponds to $\mu = \nu = \rho$ component in (3) and (116) to $\mu = \nu = i$. Other components equations in (3) vanish identically. Combining (115) and (116), we obtain

$$0 = -\frac{l^2}{4\rho^2}\left(\Phi(\phi) + \frac{12}{l^2}\right) + \frac{3}{\rho^2} + \frac{3}{F^2}\left(\partial_\rho F\right)^2 - \frac{6}{\rho F}\partial_\rho F - \frac{1}{2}\left(\partial_\rho\phi\right)^2 \tag{117}$$

$$0 = -\frac{6}{F}\partial_\rho^2 F + \frac{6}{F^2}\left(\partial_\rho F\right)^2 - \frac{6}{\rho F}\partial_\rho F - 2\left(\partial_\rho\phi\right)^2 . \tag{118}$$

If we define a new variable A, which corresponds to the exponent in the warp factor by

$$F = \rho \mathrm{e}^{2A} , \tag{119}$$

Eq.(118) can be rewritten as

$$0 = -\frac{6}{\rho}\partial_\rho\left(\rho\partial_\rho A\right) - \left(\partial_\rho\phi\right)^2 . \tag{120}$$

Now we further define a new variable B by

$$B \equiv \rho\partial_\rho A . \tag{121}$$

If $\frac{\partial\phi}{\partial\rho} \neq 0$, we can regard B as a function of ϕ instead of ρ and one obtains

$$\partial_\rho B = \frac{\partial B}{\partial\phi}\frac{\partial\phi}{\partial\rho} . \tag{122}$$

By substituting (121) and (122) into (120), we find (by assuming $\frac{\partial\phi}{\partial\rho} \neq 0$)

$$\frac{\partial B}{\partial\phi} = -\frac{1}{6\rho}\partial_\rho\phi . \tag{123}$$

Using (117) and (123) (and also (119) and (121)), we find that the dilaton equations motion (114) is automatically satisfied.

In [36], another counterterms scheme is proposed

$$S_{\rm BGM}^{(2)} = \frac{1}{16\pi G}\int d^4x\sqrt{-\hat g}\left\{\frac{6u(\phi)}{l} + \frac{l}{2u(\phi)}R\right\} , \tag{124}$$

instead of (64). Here u is obtained in terms of this paper as follows:

$$u(\phi)^2 = 1 + \frac{l^2}{12}\Phi(\phi) . \tag{125}$$

Then based on the counter terms in (124), the following expression of the trace anomaly is given in [36]:[7]

$$T = \frac{3}{2\pi G l}(-2B - u) . \tag{126}$$

The above trace anomaly was evaluated for fixed but finite ρ. If the boundary is asymptotically AdS, F in (113) goes to a constant $F \to F_0$ (F_0: a constant). Then from (119) and (121, we find the behaviors of A and B as

$$A \to \frac{1}{2}\ln\frac{F_0}{\rho} , \quad B \to -\frac{1}{2} . \tag{127}$$

Then (123) tells that the dilaton ϕ becomes a constant. Then (117) tells that

$$u = \sqrt{1 + \frac{l^2}{12}\Phi(\phi)} \to 1 . \tag{128}$$

Eqs.(127) and (128) tell that the trace anomaly (126) vanishes on the boundary. Thus, we demonstrated that trace anomaly of [36] vanishes in the UV limit what is expected also from AdS/CFT correspondence.

We should note that the trace anomaly (22) is evaluated on the boundary, i.e., in the UV limit. We evaluated the anomaly by expandind the action in the power series of ϵ in (6) and subtracting the divergent terms in the limit of $\epsilon \to 0$. If we evaluate the anomaly for finite ρ as in [36], the terms with positive power of ϵ in the expansion do not vanish and we would obtain non-vanishing trace anomaly in general. Thus, the trace anomaly obtained in this paper does not not have any contradiction with that in [36].

[7] The radial coordinate r in [36] is related to ρ by $dr = \frac{l d\rho}{4\rho}$. Therefore $\partial_r = -\frac{2\rho}{l}\partial_\rho$, especially $\partial_r A = -\frac{2\rho}{l}\partial_\rho A = -\frac{2}{l}B$.

C Remarks on boundary values

From the leading order term in the equations of motion

$$0 = -\sqrt{-\hat{G}}\frac{\partial\Phi(\phi_1,\cdots,\phi_N)}{\partial\phi_\beta} - \partial_\mu\left(\sqrt{-\hat{G}}\hat{G}^{\mu\nu}\partial_\nu\phi_\beta\right) \ , \tag{129}$$

which are given by variation of the action (130)

$$S = \frac{1}{16\pi G}\int_{M_{d+1}} d^{d+1}x\sqrt{-\hat{G}}\left\{\hat{R} - \sum_{\alpha=1}^{N}\frac{1}{2}(\hat{\nabla}\phi_\alpha)^2 + \Phi(\phi_1,\cdots,\phi_N) + 4\lambda^2\right\} \tag{130}$$

with respect to ϕ_α, we obtain

$$\frac{\partial\Phi(\phi_{(0)})}{\partial\phi_{(0)\alpha}} = 0. \tag{131}$$

The equation (131) gives one of the necessary conditions that the spacetime is asymptotically AdS. The equation (131) also looks like a constraint that the boundary value $\phi_{(0)}$ must take a special value satisfying (131) for the general fluctuations but it is not always correct. The condition $\phi = \phi_{(0)}$ at the boundary is, of course, the boundary condition, which is not a part of the equations of motion. Due to the boundary condition, not all degrees of freedom of ϕ are dynamical. Here the boundary value $\phi_{(0)}$ is, of course, not dynamical. This tells that we should not impose the equations given only by the variation over $\phi_{(0)}$. The equation (131) is, in fact, only given by the variation of $\phi_{(0)}$. In order to understand the situation, we choose the metric in the following form

$$ds^2 \equiv \hat{G}_{\mu\nu}dx^\mu dx^\nu = \frac{l^2}{4}\rho^{-2}d\rho d\rho + \sum_{i=1}^{d}\hat{g}_{ij}dx^i dx^j \ , \quad \hat{g}_{ij} = \rho^{-1}g_{ij} \ , \tag{132}$$

(If $g_{ij} = \eta_{ij}$, the boundary of AdS lies at $\rho = 0$.) and we use the regularization for the action (130) by introducing the infrared cutoff ϵ and replacing

$$\int d^{d+1}x \to \int d^dx\int_\epsilon d\rho \ , \quad \int_{M_d} d^dx\Big(\cdots\Big) \to \int d^dx\left(\cdots\right)\Big|_{\rho=\epsilon} \ . \tag{133}$$

Then the action (130) has the following form:

$$S = \frac{l}{16\pi G}\frac{1}{d}\epsilon^{-\frac{d}{2}}\int_{M_d} d^dx\sqrt{-\hat{g}_{(0)}}\left\{\Phi(\phi_{1(0)},\cdots,\phi_{N(0)}) - \frac{8}{l^2}\right\} + \mathcal{O}\left(\epsilon^{-\frac{d}{2}+1}\right) \ . \tag{134}$$

Then it is clear that Eq.(131) can be derived only from the variation over $\phi_{(0)}$ but not other components $\phi_{(i)}$ $(i = 1, 2, 3, \cdots)$. Furthermore, if we add the surface counterterm $S_b^{(1)}$

$$S_b^{(1)} = -\frac{1}{16\pi G}\frac{d}{2}\epsilon^{-\frac{d}{2}}\int_{M_d} d^d x\sqrt{-\hat{g}_{(0)}}\Phi(\phi_{1(0)}, \cdots, \phi_{N(0)}) \tag{135}$$

to the action (130), the first $\phi_{(0)}$ dependent term in (134) is cancelled and we find that Eq.(131) cannot be derived from the variational principle. The surface counterterm in (135) is a part of the surface counterterms, which are necessary to obtain the well-defined AdS/CFT correspondence. Since the volume of AdS is infinte, the action (130) contains divergences, a part of which appears in (134). Then in order that we obtain the well-defined AdS/CFT set-up, we need the surface counterterms to cancell the divergence.

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